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Frame potential for finite-dimensional Banach spaces

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ABSTRACT

We define the frame potential for a Schauder frame on a finite dimensional Banach space as the square of the 2-summing norm of the frame operator. As is the case for frames for Hilbert spaces, we prove that the frame potential can be used to characterize finite unit norm tight frames (FUNTFs) for finite dimensional Banach spaces. We prove the existence of FUNTFs for a variety of spaces, and in particular that every n-dimensional complex Banach space with a 1-unconditional basis has a FUNTF of N vectors for every $N \geq n$. However, many interesting results on FUNTFs and sums of rank-one projections for Hilbert spaces remain unknown for Banach spaces and we conclude the paper with multiple open questions.

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1. Introduction

A collection of vectors $(x_j)_{j\in I}$ in a Hilbert space H is called a *frame* if there exist constants $A, B > 0$, called the *frame bounds*, such that

$$A\|x\|^2 \leq \sum_{j\in I} |\langle x, x_j\rangle|^2 \leq B\|x\|^2 \qquad \text{for all } x \in H. \tag{1.1}$$

The frame is called *tight* if the lower frame bound A equals the upper frame bound B. Inequality (1.1) can be interpreted as stating that the map $\Theta : H \to \ell_2(I)$ given by $\Theta(x) = (\langle x, x_j\rangle)_{j\in I}$ is an isomorphic embedding. This operator allows us to define a collection of vectors $(f_j)_{j\in I}$ by $f_j = (\Theta^*\Theta)^{-1/2}x_j$ for all $j \in I$. The sequence $(f_j)_{j\in I}$ is itself a frame for H, called the *canonical dual frame* of $(x_j)_{j\in I}$, and it gives the following reconstruction formula.

$$x = \sum_{j\in I} \langle x, f_j\rangle x_j \qquad \text{for all } x \in H. \tag{1.2}$$

Thus, a frame allows for a continuous linear reconstruction formula for all vectors in a Hilbert space. We think of frames as possibly redundant coordinate systems in the sense that the vectors $(f_j)_{j\in J}$ used for reconstruction using a frame may not be unique. This redundancy can be useful in applications as, if some coefficients of a basis are lost, this results in the loss of entire dimensions, but if some frame coefficients are lost, the error can be distributed over the whole space. This concept motivates understanding which frames are most resilient to certain kinds of error [20,25]. A *finite unit norm tight frame (FUNTF)*, is a tight frame of unit vectors for a finite dimensional Hilbert space. FUNTFs are of particular interest, as they minimize both the mean squared error due to noise and the reconstruction error due to the loss of a single coefficient [20]. All orthonormal bases for a Hilbert space are equivalent, but there is a wide variety of different FUNTFs and their geometry carries multiple interesting properties.

Significant interest in FUNTFs started in 2003 when Benedetto and Fickus proved that for every $N \geq n$, every n-dimensional Hilbert space has a FUNTF of N vectors [1]. To do this they defined a positive function on the set $\{(x_j)_{j=1}^N : x_j \in H, \|x_j\| = 1\}$, called the *frame potential*, so that a collection of vectors minimizes the frame potential if and only if the vectors form a tight frame for H. Since then, researchers have been interested in understanding the geometry and topology of FUNTFs [4,11,16] and properties of the frame potential itself [17,26]. In [13], an algorithm is given to create FUNTFs and in [12] it is shown that the method of gradient descent on the frame potential can be used to create FUNTFs.

Frames have been generalized to Banach spaces in many similar but distinct ways, such as atomic decompositions [19], Banach frames [22], framings [8], and Schauder frames [7]. We will be specifically considering Schauder frames as they are a direct generalization of (1.2) without any additional assumptions. Given a finite-dimensional Banach space X

with dual X^*, a *Schauder frame* $(x_j, x_j^*)_{j=1}^N \subseteq X \times X^*$ is a sequence of pairs such that we have the following reconstruction formula

$$x = \sum_{j=1}^{N} x_j^*(x) x_j \qquad \text{for all } x \in X. \tag{1.3}$$

Comparing with (1.2), it is clear that this notion generalizes frames for Hilbert spaces. Moreover, if $(x_j)_{j=1}^n$ is a basis for X then it can be canonically considered as a Schauder frame in the following way. Let $(x_j^*)_{j=1}^n$ be the associated sequence of *biorthogonal functionals*, that is, the elements of X^* such that for $1 \leq i, j \leq n$ we have $x_j^*(x_i) = 0$ when $i \neq j$, and $x_j^*(x_j) = 1$ (this is sometimes also known as the *dual basis*, see e.g. [24, Def. 4.10]). We then have that $x = \sum_{j=1}^n x_j^*(x) x_j$ for all $x \in X$, and thus $(x_j, x_j^*)_{j=1}^n$ is a Schauder frame. This shows that in the context of finite-dimensional Banach spaces, Schauder frames are both a generalization of frames for Hilbert spaces and a generalization of bases. Moreover, this is still true in the infinite-dimensional setting as long as we consider Schauder bases, and that is why these frames are also associated with the name Schauder. It is therefore interesting to consider both what properties of frames for Hilbert spaces extend to Schauder frames, and what properties of Schauder bases extend to Schauder frames. For example, frames for Hilbert spaces can be characterized as projections of Riesz bases for Hilbert spaces [23], and Schauder frames can be characterized as projections of Schauder bases [7]. Furthermore, many of the fundamental characterizations for shrinking and boundedly complete Schauder bases extend nicely to shrinking and boundedly complete Schauder frames [3,9,10,30]. Our goal is to extend the well-developed theory and applications for finite unit norm tight frames and the frame potential to the general finite dimensional Banach space setting. In particular, we will prove that $(x_j, x_j^*)_{j=1}^N$ is a FUNTF for an n-dimensional Banach space if and only if the 2-summing norm of the frame operator of $(x_j, x_j^*)_{j=1}^N$ is $\frac{N}{\sqrt{n}}$. Though FUNTFs are relatively new mathematical objects, we show that their generalization to Banach spaces is directly connected to some classical notions in the geometry of Banach spaces.

We thank the referees for their efforts which have allowed us to improve the paper and make it more accessible.

2. Preliminaries

Let X be a finite dimensional Banach space with dual X^*. Given a sequence of pairs $(x_j, x_j^*)_{j=1}^N \subseteq X \times X^*$, the *frame operator* of $(x_j, x_j^*)_{j=1}^N$ is the map $S : X \to X$ defined by $S(x) = \sum_{j=1}^N x_j^*(x) x_j$ [18]. The sequence $(x_j, x_j^*)_{j=1}^N$ is called an *approximate Schauder frame* if the frame operator is bounded and invertible [18]. Note that a Schauder frame is an approximate Schauder frame whose frame operator is the identity. For the purpose of developing intuition we note that just as in the setting of finite-dimensional Hilbert spaces, if $\dim X = n$ and we fix a basis for X then an approximate Schauder frame corresponds to an $n \times N$ matrix Φ and an $N \times n$ matrix Ξ such that $\Phi\Xi$ is invertible.

However, we will not be using this 'matricial' point of view because we need to keep careful track of what is happening in X and what is happening in the dual space X^*.

Definition 2.1. Let X be a finite dimensional Banach space and $(x_j, x_j^*)_{j=1}^N \subseteq X \times X^*$ such that $\|x_j\| = x_j^*(x_j) = \|x_j^*\| = 1$ for all $1 \leq j \leq N$. We say that $(x_j, x_j^*)_{j=1}^N$ is a *finite unit norm tight frame* (FUNTF) for X if its frame operator is a scalar multiple of the identity.

Note that in the case where X is a Hilbert space, and X^* is identified with X in the usual manner, by the Cauchy-Schwarz inequality the condition $\|x_j\| = \langle x_j^*, x_j\rangle = \|x_j^*\| = 1$ in the definition above is equivalent to having $x_j = x_j^*$ and $\|x_j\| = 1$. Therefore, $(x_j, x_j^*)_{j=1}^N$ is a FUNTF in our definition if and only if $x_j = x_j^*$ for all $1 \leq j \leq N$ and $(x_j)_{j=1}^N$ is a FUNTF in the original Hilbert space definition.

Let us now consider Definition 2.1 in the case where the number N of pairs equals the dimension of the space X. From tracial considerations, the frame operator must then be equal to the identity. It thus follows that $(x_j)_{j=1}^N$ is a normalized basis for X, $(x_j^*)_{j=1}^N$ is the associated sequence of biorthogonal functionals, and the latter is a normalized basis for X^*: this is precisely the definition of an *Auerbach system* [24, Def. 3.29], and every finite-dimensional Banach space is known to have one such system [24, Lemma. 3.30].

The remarks above show that our notion of FUNTF generalizes the Hilbertian one, and it also generalizes the well-known concept of an Auerbach system. However, the more application-minded reader will surely wonder whether there are any practical reasons to consider such FUNTFs. We give an answer below in Proposition 6.1 where it is shown that, just as in the Hilbertian case, our FUNTFs minimize the maximal error due to the erasure of one coefficient.

Given a finite sequence of vectors $(x_j)_{j=1}^N$ in a finite dimensional Hilbert space, the *frame potential* [1] of $(x_j)_{j=1}^N$ is the value

$$FP\big((x_j)_{j=1}^N\big) = \sum_{i,j=1}^N |\langle x_j, x_i\rangle|^2. \tag{2.1}$$

The following theorem shows that this potential can be used to characterize FUNTFs for Hilbert spaces:

Theorem 2.2 *([1]). Let $(x_j)_{j=1}^N$ be a sequence of unit vectors in an n-dimensional Hilbert space with $n \leq N$. Then, the frame potential of $(x_j)_{j=1}^N$ is at least $\frac{N^2}{n}$, and $(x_j)_{j=1}^N$ is a FUNTF if and only if the frame potential of $(x_j)_{j=1}^N$ is equal to $\frac{N^2}{n}$.*

One of our main results is proving that Theorem 2.2 is also true for Schauder frames, if we generalize the notion of frame potential appropriately. The key to our approach is the fact that the frame potential can also be calculated as the square of the Hilbert-Schmidt norm of the frame operator [1, Thm. 6.1], and this in fact plays an important role in the proof of Theorem 2.2. The Hilbert-Schmidt norm is only defined for operators on Hilbert spaces, so we will make use of a more general notion that is defined for operators between Banach spaces. Recall that if $T : H_1 \to H_2$ is a bounded linear operator between Hilbert spaces, its Hilbert-Schmidt norm can be calculated as $\|T\|_{\mathrm{HS}} = \Big(\sum_i \|Te_i\|^2\Big)^{1/2}$ where (e_i) is an orthonormal basis for the domain space H_1. If instead of an orthonormal basis we consider an arbitrary finite collection of vectors $x_1, x_2, \dots, x_N \in H_1$, it is not difficult to check that we still have

$$\sum_{j=1}^{N} \|Tx_j\|^2 \le \|T\|_{\mathrm{HS}}^2 \sup_{x\in H_1, \|x\|\le 1} \sum_{j=1}^{N} |\langle x, x_j\rangle|^2,$$

and moreover it turns out that $\|T\|_{\mathrm{HS}}^2$ is the best constant we can have in the inequality above. Analogously, if X and Y are Banach spaces and $T : X \to Y$ is a linear operator, its *2-summing norm* (denoted $\pi_2(T)$) is the infimum of all constants $C \ge 0$ such that for any $x_1, x_2, \dots, x_N$ in X we have

$$\sum_{j=1}^{N} \|Tx_j\|^2 \le C^2 \sup_{x^*\in X^*, \|x^*\|\le 1} \sum_{j=1}^{N} |x^*(x_j)|^2.$$

This is indeed a generalization of the notion of being Hilbert-Schmidt, in the sense that for a linear map between Hilbert spaces its Hilbert-Schmidt norm and its 2-summing norm coincide [14, Thm. 4.10]. This naturally allows us to extend the definition of frame potential for frames of finite dimensional Hilbert spaces to Schauder frames of finite dimensional Banach spaces.

Definition 2.3. Given $(x_j, x_j^*)_{j=1}^N \subseteq X \times X^*$, we define the *frame potential* of $(x_j, x_j^*)_{j=1}^N$ as the square of the 2-summing norm of the frame operator of $(x_j, x_j^*)_{j=1}^N$.

The following result follows from Theorem 3.3 which we prove in Section 3, and is a version of Theorem 2.2 for Schauder frames.

Theorem 2.4. *Let X be an n-dimensional Banach space, $N \ge n$, $x_1, \dots, x_N \in X$ and $x_1^*, \dots x_N^* \in X^*$ such that $\|x_j\| = \|x_j^*\| = x_j^*(x_j) = 1$ for $1 \le j \le N$. Then, the frame potential of $(x_j, x_j^*)_{j=1}^N$ is at least $\frac{N^2}{n}$, and $(x_j, x_j^*)_{j=1}^N$ is a FUNTF if and only if the frame potential of $(x_j, x_j^*)_{j=1}^N$ is equal to $\frac{N^2}{n}$.*

It can be difficult to calculate the frame potential using the 2-summing norm, but there is fortunately extensive literature on the subject. Before getting into properties of the 2-summing norm and our definition of the frame potential, we will observe that the naive alternatives for an explicit simple frame potential on Banach spaces don't work. Our definition for a frame potential uses a generalization of the Hilbert-Schmidt norm, but it is conceivable that the explicit formula given in (2.1) could be generalized instead. Given $(x_j, x_j^*)_{j=1}^N \subseteq X \times X^*$, one could consider two naive candidates for an alternative formula for a frame potential given by $\sum_{j=1}^N \sum_{i=1}^N |x_j^*(x_i)|^2$ and $\sum_{j=1}^N \sum_{i=1}^N |x_j^*(x_i)x_i^*(x_j)|$. However, neither of these are appropriate formulas for a frame potential because as the following proposition shows, Theorem 2.4 would be false for both formulas. Thus, although the 2-summing norm can be difficult to calculate, it works for defining a frame potential whereas the naive alternative formulas do not.

Proposition 2.5. *Let $F_3(\ell_1^2)$ be the set of length 3 approximate Schauder frames $(x_j, x_j^*)_{j=1}^3$ of ℓ_1^2 such that $\|x_j\|_1 = \|x_j^*\|_\infty = x_j^*(x_j) = 1$ for $j = 1,2,3$. There exists a FUNTF $(x_j, x_j^*)_{j=1}^3 \in F_3(\ell_1^2)$ and $(y_j, y_j^*)_{j=1}^3, (z_j, z_j^*)_{j=1}^3 \in F_3(\ell_1^2)$ so that $(y_j, y_j^*)_{j=1}^3$ and $(z_j, z_j^*)_{j=1}^3$ are not FUNTFs, and the following two inequalities are satisfied.*

(1) $\sum_{j=1}^3 \sum_{k=1}^3 |x_j^*(x_k)|^2 > \sum_{j=1}^3 \sum_{k=1}^3 |y_j^*(y_k)|^2$
(2) $\sum_{j=1}^3 \sum_{k=1}^3 |x_j^*(x_k)x_k^*(x_j)| > \sum_{j=1}^3 \sum_{k=1}^3 |z_j^*(z_k)z_k^*(z_j)|$

Proof. Let (e_1, e_2) be the unit vector basis for ℓ_1^2 with biorthogonal functionals (e_1^*, e_2^*). We will define a FUNTF $(x_j, x_j^*)_{j=1}^3$ for ℓ_1^2 by

$$x_1 = e_1,\ x_2 = \frac{1}{4}e_1 - \frac{3}{4}e_2,\ x_3 = \frac{1}{4}e_1 + \frac{3}{4}e_2, \quad x_1^* = e_1^*,\ x_2^* = e_1^* - e_2^*,\ x_3^* = e_1^* + e_2^*.$$

Then we have that $\|x_j\|_1 = \|x_j^*\|_\infty = x_j^*(x_j) = 1$ for all $j = 1,2,3$. Furthermore, it is simple to check that for all $x \in \ell_1^2$, we have that $\sum_{j=1}^3 x_j^*(x)x_j = \frac{3}{2}x$. Thus, $(x_j, x_j^*)_{j=1}^3$ is a FUNTF for ℓ_1^2. We now define $(y_j, y_j^*)_{j=1}^3$ by

$$y_1 = e_1,\ y_2 = \frac{1}{2}e_1 - \frac{1}{2}e_2,\ y_3 = \frac{1}{2}e_1 + \frac{1}{2}e_2, \quad y_1^* = e_1^*,\ y_2^* = e_1^* - e_2^*,\ y_3^* = e_1^* + e_2^*.$$

Then we have that $\|y_j\|_1 = \|y_j^*\|_\infty = y_j^*(y_j) = 1$ for all $j = 1,2,3$. However, $\sum_{j=1}^3 y_j^*(e_1)y_j = 2e_1$ and $\sum_{j=1}^3 y_j^*(e_2)y_j = e_2$. Thus, $(y_j, y_j^*)_{j=1}^3$ is not a FUNTF.

A direct calculation shows that,

$$\sum_{j=1}^3 \sum_{k=1}^3 |x_j^*(x_k)|^2 = 5 + \frac{5}{8} > 5 + \frac{1}{2} = \sum_{j=1}^3 \sum_{k=1}^3 |y_j^*(y_k)|^2,$$

which proves (1). We now define $(z_j, z_j^*)_{j=1}^3$ by

$$z_1 = e_1,\ z_2 = e_2,\ z_3 = \frac{1}{2}e_1 + \frac{1}{2}e_2, \quad z_1^* = e_1^* - e_2^*,\ z_2^* = e_2^*,\ z_3^* = e_1^* + e_2^*.$$

Then we have that $\|z_j\|_1 = \|z_j^*\|_\infty = z_j^*(z_j) = 1$ for all $j = 1,2,3$. However, $\sum_{j=1}^3 z_j^*(e_1)z_j = \frac{3}{2}e_1 + \frac{1}{2}e_2$. Thus, $(z_j, z_j^*)_{j=1}^3$ is not a FUNTF.

A direct calculation proves (2) as,

$$\sum_{j=1}^{3}\sum_{k=1}^{3}|x_j^*(x_k)x_k^*(x_j)| = 4+\frac{1}{2} > 4 = \sum_{j=1}^{3}\sum_{k=1}^{3}|z_j^*(z_k)z_k^*(z_j)|. \qquad \square$$

3. Properties of the 2-summing norm and the frame potential

Recall that if X and Y are Banach spaces then the 2-summing norm $\pi_2(T)$ of an operator $T : X \to Y$ is the infimum of all constants $C \geq 0$ such that for any $x_1, x_2, \dots, x_N$ in X we have

$$\sum_{j=1}^{N}\|Tx_j\|^2 \leq C^2 \sup_{x^*\in X^*, \|x^*\|\leq 1}\sum_{j=1}^{N}|x^*(x_j)|^2. \tag{3.1}$$

The space of all 2-summing operators from X to Y, equipped with the norm $\pi_2(\cdot)$, will be denoted by $\Pi_2(X, Y)$ (or simply $\Pi_2(X)$ when $Y = X$). Equation (3.1) may look daunting, but the literature contains many useful techniques and results to assist us. Even though (3.1) requires us to consider sequences of any arbitrary length N, when the operator T has rank k it suffices to consider $N = k(k+1)/2$ in the real case and $N = k^2$ in the complex case [35, Thm. 18.2]. In this regard, it is worth pointing out that it is easy to see that for a rank-one operator the 2-summing norm coincides with the operator norm, from where it follows that all finite-rank operators are 2-summing. In particular, all linear operators on a finite-dimensional normed space are 2-summing (just as the same is true for Hilbert-Schmidt operators on a finite-dimensional Hilbert space). One result that we will use extensively is that when X is a finite-dimensional Banach space, the space of 2-summing operators from X to X is in trace duality with itself [35, Prop. 9.10]: in particular, for $S, T : X \to X$ linear operators, we have the following Cauchy-Schwarz type inequality

$$|\operatorname{tr}(ST)| \leq \pi_2(S)\pi_2(T).$$

Since we will only be dealing with finite-dimensional spaces, all traces in this paper have the usual linear algebra meaning: the trace of a linear operator is the sum of the diagonal elements in any matrix representation of the operator. If the operator $T : X \to X$ can be written as $T = \sum_{j=1}^{m} x_j^* \otimes x_j$ with $x_j \in X$ and $x_j^* \in X^*$ for each $1 \leq j \leq m$, note that $\operatorname{tr}(T) = \sum_{j=1}^{m} x_j^*(x_j)$: this follows from the fact that the trace is additive, and the trace of each rank-one operator $x_j^* \otimes x_j$ is easily seen to be $x_j^*(x_j)$ by considering the matrix representation of $x_j^* \otimes x_j$ with respect to any basis that contains x_j. In particular, f is a norm-1 linear functional defined on the space of 2-summing operators from X to X if and only if there is an operator $S : X \to X$ with $\pi_2(S) = 1$ and $f(T) = \operatorname{tr}(ST)$ for every operator $T : X \to X$. Another important ingredient will be the fact that if X is an n-dimensional normed space, then the 2-summing norm of the identity map

$I_X : X \to X$ is equal to $\sqrt{n}$ [14, Thm. 4.17]. We will also make use of the following characterization of 2-summing operators (see [21, Cor. 16.10.1] or [14, Cor. 2.16] for a reference).

Theorem 3.1 *(Pietsch factorization theorem). Let X and Y be Banach spaces and $T : X \to Y$ be a linear operator. Then T is a 2-summing operator if and only if there is a probability measure space M and linear operators $A : X \to L_\infty(M)$ and $B : L_2(M) \to Y$ with $\|A\|\|B\| = \pi_2(T)$ so that the following diagram commutes, where $I_{\infty,2}$ is the formal identity from $L_\infty(M)$ to $L_2(M)$.*

$$\begin{array}{ccc} L_\infty(M) & \xrightarrow[I_{\infty,2}]{} & L_2(M) \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ X & \xrightarrow[T]{} & Y \end{array}$$

Recall that a Banach space X is called *smooth* at a point $x \in X \setminus \{0\}$ if there exists a unique norming functional of x that we call x^*: that is, $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$. A Banach space is called smooth if it is smooth at every non-zero point. Note that the case of equality in the Cauchy-Schwarz inequality means that a Hilbert space is smooth. More generally, the spaces ℓ_p are smooth when $1 < p < \infty$, but not when $p = 1, \infty$. These examples give a very good picture of what smoothness means: a space X is smooth at a vector x with $\|x\| = 1$, if the unit ball of X has a unique tangent plane at x. The space of Hilbert-Schmidt operators on a Hilbert space is itself a Hilbert space, and is hence smooth. Though the space of 2-summing operators on a Banach space may not be smooth, we prove below that the space of 2-summing operators on any finite dimensional Banach space is smooth at the identity map.

Theorem 3.2. *Let X be a finite dimensional Banach space. Then the space of 2-summing operators on X is smooth at the identity map I_X.*

Proof. Let X be an n-dimensional Banach space and let $S : X \to X$ with $\pi_2(S) = 1$. We then have that

$$n^{1/2} = \pi_2(I_X) \geq \operatorname{tr}(I_X \circ S) = \operatorname{tr}(S). \tag{3.2}$$

Note that we have equality in (3.2) for $S = n^{-1/2} I_X$ and hence $n^{-1/2} I_X$ is a norming functional for I_X. We will prove that the space of 2-summing operators is smooth at the identity operator by showing that equality occurs in (3.2) if and only if $S = n^{-1/2} I_X$. Suppose that equality occurs in (3.2) for an operator S on X with $\pi_2(S) = 1$. In particular, $\operatorname{tr}(S) = n^{1/2}$. By the Pietsch factorization theorem, we have that there exists a probability measure space M and operators $A : X \to L_\infty(M)$ with $\|A\| = 1$ and $B : L_2(M) \to Y$ with $\|B\| = \pi_2(S) = 1$ so that the following diagram commutes.

$$\begin{array}{ccc} L_\infty(M) & \xrightarrow[I_{\infty,2}]{} & L_2(M) \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ X & \xrightarrow[S]{} & X \end{array}$$

Let $H = (B^{-1}(0))^\perp$ and $P_H : L_2(M) \to H$ be the orthogonal projection. Note that $\dim(H) \leq n$ as X is n-dimensional and hence $\pi_2(I_H) = \|I_H\|_{\mathrm{HS}} \leq n^{1/2}$. We now have the following inequalities

$$\begin{aligned} n^{1/2} &= \mathrm{tr}(S) = \mathrm{tr}(B|_H P_H I_{\infty,2} A) = \mathrm{tr}(P_H I_{\infty,2} AB|_H) = \mathrm{tr}(I_H P_H I_{\infty,2} AB|_H) \\ &\leq \pi_2(I_H)\pi_2(P_H I_{\infty,2} AB|_H) \leq n^{1/2}\|P_H\|\pi_2(I_{\infty,2})\|A\|\|B|_H\| \leq n^{1/2}, \end{aligned}$$

where we have used that $\pi_2(I_{\infty,2}) \leq 1$ by Pietsch factorization using the identity for A and B. Therefore, all the inequalities above are in fact equalities. In particular, $\pi_2(I_H) = n^{1/2}$ and hence B is rank n and $B|_H : H \to X$ is invertible. Furthermore,

$$\mathrm{tr}(I_H P_H I_{\infty,2} AB|_H) = \pi_2(I_H)\pi_2(P_H I_{\infty,2} AB|_H).$$

Note that H is an n-dimensional Hilbert space and the 2-summing norm coincides with the Hilbert-Schmidt norm for operators on Hilbert spaces. Thus, as the Hilbert-Schmidt norm is smooth, we have that $P_H I_{\infty,2} AB|_H = n^{-1/2} I_H$ is the unique normalizing functional of I_H. As $B|_H$ is invertible, we have that $B|_H P_H I_{\infty,2} AB|_H (B|_H)^{-1} = B|_H n^{-1/2} I_H (B|_H)^{-1}$. Hence, $S = n^{-1/2} I_X$. □

Recall that the frame potential of an approximate Schauder frame is the square of the 2-summing norm of the frame operator. Thus, the following result shows that the frame potential can be used to characterize FUNTFs.

Theorem 3.3. *Let X be an n-dimensional Banach space, $N \geq n$, $x_1, \ldots, x_N \in X$ and $(x_j, x_j^*)_{j=1}^N \subset X \times X^*$ such that $x_j^*(x_j) = 1$ for $1 \leq j \leq N$. If $S : X \to X$ is the frame operator of $(x_j, x_j^*)_{j=1}^N$, then*

$$\frac{N}{\sqrt{n}} \leq \pi_2(S). \tag{3.3}$$

Moreover, equality in (3.3) occurs if and only if $S = \frac{N}{n} I_X$.

Proof. Recall, from the general properties of the 2-summing norm stated at the beginning of this section, that $\pi_2(I_X) = \sqrt{n}$. Therefore we have the following inequality

$$\pi_2(S) = \pi_2(n^{-1/2} I_X)\pi_2(S) \geq \operatorname{tr}(n^{-1/2} I_X S) = n^{-1/2} \operatorname{tr}\left(\sum_{j=1}^{N} x_j^* \otimes x_j \right)$$
$$= n^{-1/2} \sum_{j=1}^{N} x_j^*(x_j) = n^{-1/2} N,$$

where the trace in the second-to-last equality was calculated according to the remarks at the beginning of the section. Furthermore, as the 2-summing norm is smooth at the identity operator by Theorem 3.2, we have that equality occurs if and only if S is a scalar multiple of the identity operator. Thus, there exists a scalar λ such that $S = \lambda I_X$. By taking the trace of both sides, we get that $\lambda = \frac{N}{n}$. □

4. When do FUNTFs of length N exist?

In Theorem 3.3 we have obtained a nice lower bound of N^2/n for the frame potential, which is achieved only when the associated frame operator is a multiple of the identity. Nevertheless, the result does not show that this bound is actually achieved. In the Hilbertian setting it is known that the frame potential does always achieve this bound [1, Thm. 7.1], which in particular means that in a Hilbert space we can always find FUNTFs of any given length which is at least the dimension of the space. In fact, more is true: every local minimizer of the frame potential is a FUNTF, and thus also a global minimizer [1, Thm. 7.4]. It would be very interesting to know whether either of those results is true in the Banach space setting, but at the moment we do not know the answer. We can interpret the analysis of local minimizers for the frame potential in Hilbert space in operator-theoretic terms, particularly the presentation from [6], but the arguments are very specific to the geometry of the Hilbertian setting: for example, the fact that the Hilbert-Schmidt norm satisfies the parallelogram equality.

In this section we study, independently of the frame potential, the question of the existence of FUNTFs of a given length N for a general Banach space. We start by recording the easy fact that FUNTFs always exist when their length is a multiple of the dimension of the space.

Proposition 4.1. *Let X be an n-dimensional Banach space. If N is a multiple of n, then there exists a FUNTF of length N for X.*

Proof. Let $(e_j, e_j^*)_{j=1}^n$ be an Auerbach system for X; as pointed out in Section 2, this is a FUNTF of length n for X. When N is a larger multiple of n, we simply take N/n copies of the Auerbach system. □

It should be mentioned that in general Auerbach systems are not unique. In a recent paper [36], where they prove an old conjecture of Pełczyński, the authors Weber and Wojciechowski show that a Banach space of dimension $n > 2$ has at least $(n-1)n/2 + 1$ Auerbach systems.

Note that if $y \in X$ and $y^* \in X^*$ satisfy $1 = \|y\| = \|y^*\| = y^*(y)$ then the operator $x \mapsto y^*(x)y$ is a norm-one, rank-one projection, and in fact this characterizes the norm-one, rank-one projections. Therefore, looking for FUNTFs corresponds to figuring out when a multiple of the identity can be written as a sum of norm-one, rank-one projections. In the Hilbert space case, invertible operators that can be written as such sums have a very nice characterization [28]: they need to be positive and have integer trace at least the dimension of the space. For complex Banach spaces, a related question (dropping the norm one condition) also has a very satisfactory answer: [2, Thm. 4.4] shows that an operator $T : X \to X$ is a sum of rank-one projections if and only if $\mathrm{tr}(T)$ is an integer and $\mathrm{rank}(T) \leq \mathrm{tr}(T)$.

We have not been able to obtain, for a general finite-dimensional Banach space X, a characterization of the invertible operators $X \to X$ that can be written as a sum of N norm-one, rank-one projections. Nevertheless, we prove below that in the special case of a diagonal operator on a complex space admitting a special type of basis, the "obvious" conditions are enough. As a consequence, we prove the existence of FUNTFs on such spaces (and a few others). The crux of the argument is given by the following lemma, where the number of projections is equal to the dimension of the space. Recall that a basis $(e_j)_{j=1}^n$ is called *1-unconditional if* whenever $(\alpha_j)_{j=1}^n, (\beta_j)_{j=1}^n$ are scalars satisfying $|\alpha_j| \leq |\beta_j|$ for each $1 \leq j \leq n$, we have $\big\| \sum_{j=1}^n \alpha_j e_j \big\| \leq \big\| \sum_{j=1}^n \beta_j e_j \big\|$. Orthonormal bases are 1-unconditional, and so are the canonical bases in the spaces ℓ_p^n. For the convenience of readers who may not be familiar with this concept, we prove below an elementary fact.

Lemma 4.2. *Let X be an n-dimensional Banach space with a 1-unconditional basis $(e_j)_{j=1}^n$ and corresponding biorthogonal functionals $(e_j^*)_{j=1}^n$.*

(a) *$(e_j^*)_{j=1}^n$ is also 1-unconditional, and is normalized when $(e_j)_{j=1}^n$ is normalized.*
(b) *Let $y = \sum_{j=1}^n y_j e_j$ with $y_j \geq 0$ for each $1 \leq j \leq n$. Then there exist $z_j \geq 0$, $1 \leq j \leq n$ such that $z^* = \sum_{j=1}^n z_j e_j^*$ satisfies $\|z^*\| = 1$ and $z^*(y) = \|y\|$.*

Proof. (a) Let $(\alpha_j)_{j=1}^n, (\beta_j)_{j=1}^n$ be scalars satisfying $|\alpha_j| \leq |\beta_j|$ for each $1 \leq j \leq n$. Since X is finite-dimensional, there exists $x = \sum_{j=1}^n x_j e_j$ such that $\|x\| = 1$ and $\big(\sum_{j=1}^n \alpha_j e_j^*\big)(x) = \big\| \sum_{j=1}^n \alpha_j e_j^* \big\|$. For each $1 \leq j \leq n$, let $\theta_j = \alpha_j / \beta_j$ when $\beta_j \neq 0$, and $\theta_j = 0$ when $\beta_j = 0$. Since (e_j) is 1-unconditional, $\big\| \sum_{j=1}^n \theta_j x_j e_j \big\| \leq \|x\| = 1$. Therefore,

$$\Big\| \sum_{j=1}^n \beta_j e_j^* \Big\| \geq \Big(\sum_{j=1}^n \beta_j e_j^* \Big) \Big(\sum_{j=1}^n \theta_j x_j e_j \Big) = \sum_{j=1}^n \beta_j \theta_j x_j = \sum_{j=1}^n \alpha_j x_j$$
$$= \Big(\sum_{j=1}^n \alpha_j e_j^* \Big)(x) = \Big\| \sum_{j=1}^n \alpha_j e_j^* \Big\|,$$

which shows that $(e_j^*)_{j=1}^n$ is unconditional. Assume now that $\|e_j\| = 1$ for each $1 \le j \le n$. Since $(e_j)_{j=1}^n$ is 1-unconditional, for any scalars $(x_j)_{j=1}^n$ we have $\left\|\sum_{j=1}^n x_j e_j\right\| \ge \|x_1 e_1\| = |x_1|$. Therefore, $\left|e_1^*\left(\sum_{j=1}^n x_j e_j\right)\right| = |x_1| \le \left\|\sum_{j=1}^n x_j e_j\right\|$, so $\|e_1^*\| \le 1$. Since $e_1^*(e_1) = 1$ and $\|e_1\| = 1$, it follows that $\|e_1^*\| = 1$. That $\left\|e_j^*\right\| = 1$ for all $1 \le j \le n$ follows analogously.

(b) Let $y^* = \sum_{j=1}^n \gamma_j e_j^*$ satisfy $\|y^*\| = 1$ and $\|y\| = y^*(y)$. Let $z_j = |\gamma_j|$ and set $z^* = \sum_{j=1}^n z_j e_j^*$. By part (a), $\|z^*\| = \|y^*\| = 1$. Therefore,

$$\|y\| = y^*(y) = \sum_{j=1}^n \gamma_j y_j \le \sum_{j=1}^n |\gamma_j| y_j = z^*(y) \le \|z^*\| \, \|y\| = \|y\| . \qquad \square$$

The following lemma is a particular case of a factorization theorem due to Lozanovskiĭ [31]. The specific finite-dimensional version below is a restatement of [27, Main Lemma], keeping in mind that their terminology is different: a normalized 1-unconditional basis corresponds to what they call a *balanced norm*. Since [27] only works with real scalars, for the convenience of the reader we provide a sketch of the proof that also allows for complex scalars. In a nutshell, the lemma says that if X is n-dimensional and has a normalized 1-unconditional basis, then any nonnegative element of ℓ_1^n can be 'factored as a product' of a norm-one element in X and a norm-one element in X^*.

Lemma 4.3. *Let X be an n-dimensional Banach space with a normalized 1-unconditional basis $(e_j)_{j=1}^n$, and corresponding biorthogonal functionals $(e_j^*)_{j=1}^n$. For any sequence of nonnegative numbers $(\lambda_j)_{j=1}^n$ with $\sum_{j=1}^n \lambda_j = 1$, there exist sequences of nonnegative numbers $(\alpha_j)_{j=1}^n, (\beta_j)_{j=1}^n$ such that both $x = \sum_{j=1}^n \alpha_j e_j$ and $x^* = \sum_{j=1}^n \beta_j e_j^*$ have norm one, and moreover $\alpha_j \beta_j = \lambda_j$ for each $1 \le j \le n$.*

Sketch of proof. We will assume that X is smooth; the general case follows by approximation as in [27]. Define the *duality map* $J : X \to X^*$ as follows: for any $x \in X$, Jx is the unique functional in X^* satisfying $\|Jx\| = \|x\|$ and $(Jx)(x) = \|x\|^2$. It is not hard to see that J is a continuous function. We will denote the nonnegative part of the unit sphere of ℓ_1^n by

$$S_n^+ = \left\{ (x_1, \dots x_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^n x_j = 1 \text{ and } x_j \ge 0 \text{ for each } 1 \le j \le n \right\}.$$

Note that S_n^+ is an $(n-1)$-simplex.

For $n = 1$ the lemma is obvious, so let us assume $n = 2$. The conclusion is clear when (λ_1, λ_2) is equal to either $(1, 0)$ or $(0, 1)$, we simply take $x = e_1, x^* = e_1^*$ or $x = e_2, x^* = e_2^*$ respectively (note that $\|e_1^*\| = \|e_2^*\| = 1$ by Lemma 4.2). Let $\alpha = (\alpha_1, \alpha_2) : [0, 1] \to (\mathbb{R}^+)^2$ be a continuous function with $\alpha(0) = (1, 0)$, $\alpha(1) = (0, 1)$ and $\|x(t)\| = 1$ for each $t \in [0, 1]$, where $x(t) = \alpha_1(t) e_1 + \alpha_2(t) e_2$. Since the duality map J is continuous, if we write $J(x(t)) = \beta_1(t)e_1^* + \beta_2(t)e_2^*$ it follows that β_1 and β_2 are continuous functions. Note that from Lemma 4.2 and the smoothness assumption, $\beta_1, \beta_2 \geq 0$ because $\alpha_1, \alpha_2 \geq 0$. Therefore, the map $\Lambda : [0,1] \to S_2^+$ given by $t \mapsto \big(\alpha_1(t)\beta_1(t), \alpha_2(t)\beta_2(t)\big)$ is a continuous curve joining $(1,0)$ and $(0,1)$ within S_2^+. Since the latter is a line segment, it follows that Λ must be surjective (if any point were missing, we would have a disconnected continuous image of a connected set).

The proof for larger values of n is very close in spirit to the one above, though it requires topological tools of a more advanced nature. Assume that the lemma has been proved for n-dimensional spaces, and let X be an $(n+1)$-dimensional space satisfying the hypothesis of the lemma. Define

$$S_X^+ = \left\{ (x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \left\| \sum_{j=1}^{n+1} x_j e_j \right\| = 1 \text{ and } x_j \geq 0 \text{ for each } 1 \leq j \leq n+1 \right\}.$$

Note that S_X^+ is homeomorphic to S_{n+1}^+: for example, by taking a radial projection $S_{n+1}^+ \to S_X^+$. By the continuity of J and as in the proof above, we construct a continuous map $\Lambda : S_{n+1}^+ \to S_{n+1}^+$ which, by the induction hypothesis, maps every face of the simplex S_{n+1}^+ onto itself. By [27, Lemma 2.1], it follows that the map Λ is surjective. □

We now show that in complex spaces with a normalized 1-unconditional basis, certain diagonal operators can be written as a sum of norm-one, rank-one projections.

Lemma 4.4. *Let X be a complex n-dimensional Banach space (or a 2-dimensional real space) with a normalized 1-unconditional basis $(e_j)_{j=1}^n$ and corresponding biorthogonal functionals $(e_j^*)_{j=1}^n$. For any sequence of nonnegative numbers $(\lambda_j)_{j=1}^n$ with $\sum_{j=1}^n \lambda_j = n$, the operator $T : X \to X$ given by*

$$T = \sum_{j=1}^{n} \lambda_j e_j^* \otimes e_j$$

can be written as a sum of n norm-one, rank-one projections.

Proof. The following argument is inspired by the discrete Fourier transform, and is related to similar constructions in the Hilbert space case [20,37]. We assume first that the space is complex. Note that the nonnegative numbers λ_j/n add up to one, so by Lemma 4.3 there exist sequences of nonnegative numbers $(\alpha_j)_{j=1}^n$ and $(\beta_j)_{j=1}^n$ such that both $x = \sum_{j=1}^n \alpha_j e_j$ and $x^* = \sum_{j=1}^n \beta_j e_j^*$ have norm one, and $\alpha_j\beta_j = \lambda_j/n$ for each $1 \leq j \leq n$. Let $\omega_n = e^{-2\pi i/n}$ and for $0 \leq k \leq n-1$, let

$$x_k = \sum_{j=1}^{n} \omega_n^{kj} \alpha_j e_j \quad \text{and} \quad x_k^* = \sum_{j=1}^{n} \omega_n^{-kj} \beta_j e_j^*.$$

Note that both x_k and x_k^* have norm one by the 1-unconditionality of $(e_j)_{j=1}^n$, and moreover $x_k^*(x_k) = \sum_{j=1}^n \alpha_j\beta_j = 1$. Writing the map $x_k^* \otimes x_k$ as a matrix with respect to the bases $(e_j)_{j=1}^n$ and $(e_j^*)_{j=1}^n$, its entry in the (i,j) position is

$$\omega_n^{ki}\alpha_i\omega_n^{-kj}\beta_j$$

and therefore the entry of $\sum_{k=0}^{n-1} x_k^* \otimes x_k$ in the (i,j) position is

$$\begin{cases} \sum_{k=0}^{n-1} \alpha_j\beta_j = n\lambda_j/n = \lambda_j, & \text{when } i=j, \\ \alpha_i\beta_j \sum_{k=0}^{n-1} \omega_n^{k(i-j)} = 0, & \text{when } i \neq j. \end{cases}$$

That is, $\sum_{j=1}^n \lambda_j e_j^* \otimes e_j = \sum_{k=0}^{n-1} x_k^* \otimes x_k$.

A very similar argument works in the real case for $n = 2$: take $x_1 = \alpha_1 e_1 + \alpha_2 e_2$, $x_2 = \alpha_1 e_1 - \alpha_2 e_2$, $x_1^* = \beta_1 e_1 + \beta_2 e_2$ and $x_2^* = \beta_1 e_1 - \beta_2 e_2$; it follows that

$$x_1^* \otimes x_1 + x_2^* \otimes x_2 = \lambda_1 e_1^* \otimes e_1 + \lambda_2 e_2^* \otimes e_2. \qquad \square$$

Now we can prove the existence of FUNTFs in a wide variety of spaces.

Proposition 4.5. *Let X be a complex n-dimensional Banach space (or a 2-dimensional real space) with a normalized 1-unconditional basis $(e_j)_{j=1}^n$ and corresponding biorthogonal functionals $(e_j^*)_{j=1}^n$. Let $N \geq n$ be an integer, and assume the nonnegative numbers $(\lambda_j)_{j=1}^n$ satisfy $\sum_{j=1}^n \lambda_j = N$. Then the operator $T : X \to X$ given by*

$$T = \sum_{j=1}^n \lambda_j e_j^* \otimes e_j$$

can be written as a sum of N norm-one, rank-one projections. In particular, there exists a FUNTF of length N for X.

Proof. We will proceed by induction on N. If $N = n$, we're done by Lemma 4.4. Now suppose the statement holds whenever the operator has trace N, and take a sequence of positive numbers $(\lambda_j)_{j=1}^n$ adding up to $N+1$. Note that there exists λ_{j_0} strictly greater than one, since $n < N+1$. Consider now the sequence $(\lambda_j')_{j=1}^n$ where λ_{j_0} is replaced by $\lambda_{j_0} - 1$; the corresponding operator $T' = \sum_{j=1}^n \lambda_j' e_j^* \otimes e_j$ can be expressed as a sum of N rank-one, norm-one projections; adding $e_{j_0}^* \otimes e_{j_0}$ gives a corresponding decomposition for $T = \sum_{j=1}^n \lambda_j e_j^* \otimes e_j$. $\square$

Putting together our various results, this is the most general setting where the frame potential can be used to find tight unit norm frames.

Theorem 4.6. *Let X be a complex n-dimensional Banach space (or a 2-dimensional real space) with a normalized 1-unconditional basis. A sequence $(x_j, x_j^*)_{j=1}^N$ in $X \times X^*$*

satisfying $\|x_j\| = x_j^*(x_j) = \|x_j^*\| = 1$ *for each* $1 \le j \le N$ *minimizes the 2-summing norm of its associated frame operator if and only if it is a FUNTF.*

Proof. According to Theorem 3.3, all we need to do is show the existence of one such sequence whose frame operator has 2-summing norm exactly $\frac{N}{\sqrt{n}}$; this is a consequence of Proposition 4.5 applied to the operator $\frac{N}{n}I_X$. □

5. Smoothness and strict convexity of $\Pi_2(X, X)$

In Theorem 3.2, we proved that the space of 2-summing operators $\Pi_2(X, X)$, is smooth at I_X for any finite-dimensional Banach space X. Setting our aim higher, it would be interesting to know when $\Pi_2(X, X)$ is itself smooth at every point. Notice that X^* embeds isometrically into $\Pi_2(X, X)$: it is easy to see that for $x \in X$ and $x^* \in X^*$ the rank-one operator $x^* \otimes x$ is 2-summing with $\pi_2(x^* \otimes x) = \|x^*\| \cdot \|x\|$, and thus if we fix $x_0 \in X$ with $\|x_0\| = 1$ the mapping $x^* \mapsto x^* \otimes x_0$ gives an isometric embedding of X^* into $\Pi_2(X, X)$. Therefore X being strictly convex will be a necessary condition for smoothness of $\Pi_2(X, X)$, as strict convexity and smoothness are dual properties. Moreover $\Pi_2(X, X)$ is in trace duality with itself, so we may equivalently study when $\Pi_2(X, X)$ is strictly convex. Recall that a Banach space Y is called *strictly convex* if whenever $x, y \in Y$ are such that $\|x\| = \|y\| = \frac{1}{2}\|x + y\|$ we have that $x = y$. The following result gives a characterization of when the space of 2-summing operators is strictly convex in a slightly more general setting, and is done in terms of a unique-extension condition for 2-summing maps on subspaces.

Lemma 5.1. *Let X and Y be finite-dimensional Banach spaces, with Y being strictly convex. The following conditions are equivalent:*

(a) $\Pi_2(X, Y)$ *is strictly convex.*
(b) *For any subspace E of X and any linear operator $t : E \to Y$, there exists a unique linear extension $T : X \to Y$ with $\pi_2(T) = \pi_2(t)$.*

Proof. (b) ⇒ (a): Let $T, S : X \to Y$ be linear operators with $\pi_2(T) = \pi_2(S) = \pi_2\big(\frac{1}{2}(S + T)\big) = 1$. Let $(x_j)_{j=1}^N$ be a collection of vectors in X such that

$$\sup_{x^* \in X^*, \|x^*\| \le 1} \sum_{j=1}^N |x^*(x_j)|^2 = 1 \quad \text{and} \quad \pi_2\big(\tfrac{1}{2}(S + T)\big) = \Big(\sum_{j=1}^N \big\|\tfrac{1}{2}(S + T)x_j\big\|^2\Big)^{1/2}.$$

Note that

$$\begin{aligned}\pi_2\big(\tfrac{1}{2}(S + T)\big) &= \big\|(\tfrac{1}{2}\|Sx_j + Tx_j\|)_{j=1}^N\big\|_{\ell_2} \le \big\|(\tfrac{1}{2}\|Sx_j\| + \tfrac{1}{2}\|Tx_j\|)_{j=1}^N\big\|_{\ell_2} \\ &= \big\|\tfrac{1}{2}(\|Sx_j\|)_{j=1}^N + \tfrac{1}{2}(\|Tx_j\|)_{j=1}^N\big\|_{\ell_2}\end{aligned}$$

$$\leq \tfrac{1}{2}\left\|(\|Sx_j\|)_{j=1}^N\right\|_{\ell_2} + \tfrac{1}{2}\left\|(\|Tx_j\|)_{j=1}^N\right\|_{\ell_2}$$
$$\leq \tfrac{1}{2}\pi_2(S) + \tfrac{1}{2}\pi_2(T).$$

Since we have that $\pi_2\big(\frac{1}{2}(S+T)\big) = \frac{1}{2}\pi_2(S) + \frac{1}{2}\pi_2(T)$, all the inequalities above must be equalities and it follows that $Tx_j = Sx_j$ for $1 \leq j \leq N$. Let $E = \operatorname{span}(x_j)_{j=1}^N \subset X$, and let $v : E \to Y$ be the restriction of T (or S) to E. By the choice of the sequence $(x_j)_{j=1}^N$, note that $\pi_2(v) = 1$. By assumption, there is a unique extension of v to an operator $V : X \to Y$ with $\pi_2(V) = 1$. Since both S and T are extensions of v with 2-summing norm equal to 1, it follows that $S = T = V$.

(a) $\Rightarrow$ (b): We will prove the contrapositive. Suppose there exist a subspace $E \subseteq X$ and an operator $t : E \to Y$ admitting two distinct extensions $S, T : X \to Y$ with $\pi_2(t) = \pi_2(S) = \pi_2(T)$. By homogeneity, we may assume $\pi_2(t) = 1$. Note that $\frac{1}{2}(S+T)$ is also an extension of t, and therefore

$$1 = \pi_2(t) \leq \pi_2\big(\tfrac{1}{2}(S+T)\big) \leq \tfrac{1}{2}\pi_2(S) + \tfrac{1}{2}\pi_2(T) = 1,$$

so $\pi_2\big(\frac{1}{2}(S+T)\big) = 1$, showing that $\Pi_2(X,Y)$ is not strictly convex. □

It is a well-known and important property of 2-summing maps that if $t : E \to Y$ is 2-summing, and E is a subspace of X, then there exists an extension $T : X \to Y$ with $\pi_2(T) = \pi_2(t)$ (this follows easily from the Pietsch factorization theorem and the 1-injectivity of L_∞ spaces, see [14, Thm. 4.15]). Lemma 5.1 above shows that the uniqueness of such extensions is related to geometric properties of the space of 2-summing maps, and is of the same nature as the following classical theorem due to Taylor [34] and Foguel [15]:

Theorem 5.2. *For a normed space X, the following conditions are equivalent:*

(a) *X^* is strictly convex.*
(b) *For any subspace E of X and any linear functional $t : E \to \mathbb{K}$, there exists a unique linear extension $T : E \to \mathbb{K}$ with $\|T\| = \|t\|$.*

There are results related to Theorem 5.2 that give equivalent geometrical characterizations of unique extension properties (not only the general situation above, but also specializations to extensions for a given fixed subspace $E \subseteq X$ or even for a given fixed functional $t : E \to \mathbb{K}$; see for example [5] and [32]). Putting together Lemma 5.1 and [32, Thm. 3.1], we get the following:

Corollary 5.3. *Let X and Y be finite-dimensional Banach spaces, with Y being strictly convex. The following conditions are equivalent:*

(a) $\Pi_2(X,Y)$ *is strictly convex.*
(b) *For any subspace E of X and any linear operator $t : E \to Y$, there exists a unique linear extension $T : E \to Y$ with $\pi_2(T) = \pi_2(t)$.*
(c) *For any subspace E of X, any $\varepsilon > 0$, any $T \in \Pi_2(Y,X)$ and any sequence (S_n) in $\Pi_2(Y,E)$ with $\pi_2(S_1) \leq 1$ and $\pi_2(S_{n+1} - S_n) \leq 1$ for all $n \in \mathbb{N}$, there exist $S \in \Pi_2(Y,E)$ and $n_0 \in \mathbb{N}$ such that*

$$\pi_2(T - S \pm S_{n_0}) \leq n_0 + \varepsilon.$$

Proof. Notice that $\Pi_2(E,Y) = \big(\Pi_2(Y,E)\big)^*$ and $\Pi_2(X,Y) = \big(\Pi_2(Y,X)\big)^*$ via trace duality in both cases. Since $\Pi_2(Y,E)$ is isometrically contained in $\Pi_2(Y,X)$ in the obvious way, condition (b) is just a particular case of the uniqueness of extensions for linear functionals characterized in [32, Thm. 3.1]. □

In the infinite-dimensional case, the question of the uniform convexity of $\Pi_2(X,X)$ has been studied up to isomorphism. Lin has shown that if $\Pi_2(X,Y)$ is B-convex (in particular, if it is superreflexive), then both X and Y have cotype $2+\varepsilon$ for any $\varepsilon > 0$ [29]. Additionally, if E is superreflexive and has cotype 2 then $\Pi_2(\ell_2, E)$ is superreflexive as well. This is done with an ultraproduct argument, and in fact it follows from the following: $\Pi_2(\ell_2^n, E)$ is isomorphic to a subspace of $L_2(E)$ when E has cotype 2, and the Banach-Mazur distance between the two spaces is less than $2C_2(E)$ (where $C_2(E)$ is the cotype 2 constant of E). Pisier has proved a closely related result [33], namely that $\Pi_2(\ell_p, \ell_p)$ is superreflexive when $1 < p < 2$. The argument uses complex interpolation, and explicitly what is shown is that $\Pi_2(\ell_p, \ell_p)$ has an equivalent norm that is strictly convex.

In the 2-dimensional case, we can prove that one always has uniqueness of extensions preserving the 2-summing norm.

Proposition 5.4. *Let X be a 2-dimensional, strictly convex and smooth space. Then $\Pi_2(X,X)$ is strictly convex.*

Proof. By Lemma 5.1, it suffices to show the uniqueness of extensions preserving the 2-summing norm. Since X is 2-dimensional, it suffices to consider 1-dimensional subspaces. So let $E \subset X$ be a 1-dimensional subspace, and $t : E \to X$ be a non-zero operator. Since t has rank one we have $\|t\| = \pi_2(t)$. Note that any extension T of t preserving the 2-summing norm will also have 2-summing norm equal to its norm, since

$$\pi_2(T) \geq \|T\| \geq \|t\| = \pi_2(t).$$

There is an easy way to construct an extension that preserves the 2-summing norm: since every 1-dimensional subspace is 1-complemented, there is a norm-one projection $P : X \to E$. Note that then $S = t \circ P$ is an extension of t, clearly $\pi_2(S) \geq \pi_2(t)$ by virtue of being an extension, and $\pi_2(S) = \pi_2(t \circ P) \leq \pi_2(t)\,\|P\| = \pi_2(t)$ so $\pi_2(S) = \pi_2(t)$.

We now assume that $T : X \to X$ is a different extension of t such that $\pi_2(T) = 1$. Choose $f \in S_{X^*}$ and $y \in S_X$ such that $t(x) = f(x)y$ for all $x \in E$. Choose $x_0 \in f^{-1}(0)$ with $\|x_0\| = 1$ and choose $x_1 \in S_E$ so that $f(x_1) = 1$. We have that $T(x_0) \neq 0$ because otherwise we would have $T(ax_0 + bx_1) = t(bx_1) = (t \circ P)(ax_0 + bx_1)$ for all scalars a, b. Let $g \in S_{X^*}$ such that $g(y) = \|y\|$. As X is uniformly smooth and $f \in S_{X^*}$ is the unique normalizing functional of x_1, we have that

$$\lim_{a\to 0} \frac{\|x_1 + ax_0\| - 1}{a} = f(x_0) = 0.$$

Thus, $\forall \epsilon > 0$, $\exists \delta_\epsilon > 0$ so that $\|x_1 + ax_0\| < 1 + a\epsilon$ for all $0 < a < \delta_\epsilon$. For the sake of contradiction we assume that $g(T(x_0)) \neq 0$ and without loss of generality that $g(T(x_0)) > 0$. Then for $0 < a < \delta_{g(T(x_0))}$ we have that

$$\|T(x_1 + ax_0)\| \geq g(T(x_1) + aT(x_0)) = 1 + ag(T(x_0)) > \|x_1 + ax_0\|.$$

This contradicts that $\|T\| = 1$. Thus, we must have that $g(T(x_0)) = 0$. As $\pi_2(T) = 1$, we have that for all $0 < a$ that there exists $x_a^* \in S_{X^*}$ such that

$$1 + a^2\|T(x_0)\|^2 = \|T(x_1)\|^2 + \|T(ax_0)\|^2 \leq |x_a^*(x_1)|^2 + a^2|x_a^*(x_0)|^2.$$

By taking the limit $a \to 0$ we have that $1 \leq \lim_{a\to 0} |x_a^*(x_1)| \leq 1$ and hence as every unit norm functional that normalizes x_1 in absolute value is of the form ϵf for some $|\epsilon| = 1$, we have without loss of generality that $\lim_{a\to 0} x_a^* = f$. We now have that for all $a > 0$

$$1 + a^2\|T(x_0)\|^2 \leq |x_a^*(x_1)|^2 + a^2|x_a^*(x_0)|^2 \leq 1 + a^2|x_a^*(x_0)|^2,$$

from where it follows that $\|T(x_0)\|^2 \leq |x_a^*(x_0)|^2$. As $T(x_0) \neq 0$, this contradicts that $\lim_{a\to 0} |x_a^*(x_0)| = |f(x_0)| = 0$. □

If X is isometric to a Hilbert space then $\Pi_2(X, X)$ is isometric to a Hilbert space as well, and is hence strictly convex. In contrast to this, the following result shows that $\Pi_2(X, X)$ fails strict convexity when X is not isometric to a Hilbert space but has a non-1-complemented 2-dimensional subspace that is.

Theorem 5.5. *If ℓ_2^2 is isometric to a subspace of X that is not 1-complemented in X, then $\Pi_2(X, X)$ is not strictly convex.*

Proof. Let $Y \subseteq X$ be isometric to ℓ_2^2 and not be 1-complemented in X. Let (e_1, e_2) be an orthonormal basis for Y with biorthogonal functionals (e_1^*, e_2^*) in X^* such that $\|e_1^*\| = \|e_2^*\| = 1$. If we consider the operator $P_1 : X \to Y \subseteq X$ given by $P_1(x) = e_1^*(x)e_1 + e_2^*(x)e_2$ then P_1 is a projection onto Y with $\pi_2(P_1) = \sqrt{2} = \pi_2(I_Y)$.

As Y is not 1-complemented, there exists $y \in X$ such that $\|y\| < \|P_1(y)\| = 1$. We now choose a new orthonormal basis (f_1, f_2) for Y with biorthogonal functionals (f_1^*, f_2^*)

in X^* such that $f_1 = P_1(y)$ and $\|f_1^*\| = \|f_2^*\| = 1$. The operator $P_2 : X \to Y \subseteq X$ given by $P_2(x) = f_1^*(x)f_1 + f_2^*(x)f_2$ is a projection onto Y with $\pi_2(P_2) = \sqrt{2} = \pi_2(I_Y)$. For the sake of contradiction, we assume that $P_1 = P_2$. Then, $P_2(y) = P_1(y) = f_1$. Thus we have that $f_1^*(y) = 1$ and $f_2^*(y) = 0$. Hence, $\|P_2(y)\| = |f_1^*(y)| \le \|y\|$. This contradicts that $P_2 = P_1$. We thus have two different linear extensions of $I|_Y$ with $\pi_2(P_1) = \pi_2(P_2) = \pi_2(I|_Y)$ and hence $\Pi_2(X, X)$ is not strictly convex. □

6. Optimal frames for erasures

We will now prove a result in the spirit of Holmes and Paulsen [25], by introducing a numerical measure of how well a frame reconstructs vectors when one or more of the frame coefficients of a vector is lost. Let $(x_j, x_j^*)_{j=1}^N$ be a Schauder frame. Fix $1 \le k_1 < \cdots < k_m \le N$, and suppose that the $k_1, ..., k_m$ frame coefficients (that is, the measurements corresponding to $x_{k_1}^*, ..., x_{k_m}^*$) are lost. Let $S_{[k_1,\ldots,k_m]}$ be the frame operator associated to the situation with the lost coefficients. That is, $S_{[k_1,\ldots,k_m]} = \sum_{i \ne k_1,\ldots,k_m} x_i^* \otimes x_i$. Define the *maximal erasure error* for the frame $(x_j, x_j^*)_{j=1}^N$ due to the loss of m coordinates to be

$$e_m\big((x_j, x_j^*)_{j=1}^N\big) = \max_{1 \le k_1 < \cdots < k_m \le N} \left\| S - S_{[k_1,\ldots,k_m]} \right\| .$$

Note that in the case of the loss of one coordinate we have that

$$e_1\big((x_j, x_j^*)_{j=1}^N\big) = \max_{1 \le j \le N} \left\| S - S_{[j]} \right\| = \max_{1 \le j \le N} \|x_j^* \otimes x_j\| = \max_{1 \le j \le N} \|x_j^*\| \|x_j\|.$$

In the case when X is a finite dimensional Hilbert space, the equal norm tight frames minimize the erasure error due to the loss of one coordinate [20], and the equiangular frames (when they exist) minimize the erasure error due to the loss of two coordinates [25]. In the case of one erasure, we have the corresponding result for Banach spaces.

Proposition 6.1. *Let $N \ge n$ and let X be an n-dimensional Banach space such that there exists a FUNTF for X of length N. Suppose that $(x_j, x_j^*)_{j=1}^N$ is a Schauder frame for X. Then the following are equivalent:*

(a) $(x_j, x_j^*)_{j=1}^N$ *minimizes the maximal error due to one erasure;*
(b) $\|x_j\| \|x_j^*\| = x_j^*(x_j) = n/N$ *for all* $1 \le j \le N$*;*
(c) $\big(\frac{x_j}{\|x_j\|}, \frac{x_j^*}{\|x_j^*\|}\big)_{j=1}^N$ *is a FUNTF.*

Proof. We first prove $(b) \Rightarrow (c)$. Suppose $\|x_j\| \|x_j^*\| = x_j^*(x_j) = n/N$ for all $1 \le j \le N$. For any constant d and nonzero constants $c_1, ..., c_N$ we have that $(dc_j x_j, \frac{1}{c_j} x_j^*)_{j=1}^N$ has a frame operator of d times the frame operator of $(x_j, x_j^*)_{j=1}^N$. Thus, the frame operator of $\big(\frac{x_j}{\|x_j\|}, \frac{x_j^*}{\|x_j^*\|}\big)_{j=1}^N$ is $\frac{N}{n} I_X$. Furthermore, $\|\frac{x_j}{\|x_j\|}\| = \|\frac{x_j^*}{\|x_j^*\|}\| = \frac{x_j^*}{\|x_j^*\|}\big(\frac{x_j}{\|x_j\|}\big) = 1$ for all

$1 \leq j \leq N$. Thus $(\frac{x_j}{\|x_j\|}, \frac{x_j^*}{\|x_j^*\|})_{j=1}^N$ is a FUNTF. The argument can be reversed to show $(c) \Rightarrow (b)$ as any FUNTF has frame operator $\frac{N}{n} I_X$.

We now prove $(a) \Rightarrow (b)$ by contrapositive. We assume that it is not the case that $\|x_j\|\|x_j^*\| = x_j^*(x_j) = n/N$ for all $1 \leq j \leq N$. As $(x_j, x_j^*)_{j=1}^N$ is a Schauder frame, its frame operator is $\sum_{j=1}^N x_j^* \otimes x_j = I_X$. By taking the trace, we have that $\sum_{j=1}^N x_j^*(x_j) = n$. Thus,

$$\sum_{j=1}^N \|x_j\|\|x_j^*\| \geq \sum_{j=1}^N x_j^*(x_j) = n.$$

Hence there exists $1 \leq k \leq N$ such that $\|x_k\|\|x_k^*\| > \frac{n}{N}$. We have that $\|x_k^* \otimes x_k\| = \|x_k^*\|\|x_k\| > n/N$ is the error due to the erasure of the kth coordinate. However, there exists a FUNTF $(y_j, y_j^*)_{j=1}^N$ of X. Thus, $(\frac{n}{N} y_j, y_j^*)_{j=1}^N$ is a Schauder frame of X and $\frac{n}{N}$ is the error due to one erasure. Thus, $(x_j, x_j^*)_{j=1}^N$ does not minimize the maximal error due to one erasure.

We now prove $(b) \Rightarrow (a)$. We have previously shown that if $(x_j, x_j^*)_{j=1}^N$ minimizes the maximal error due to one erasure then it satisfies (b). However, any frame that satisfies (b) has the same error due to one erasure of exactly n/N. Thus, if $(x_j, x_j^*)_{j=1}^N$ satisfies (b) then it minimizes the maximal error due to one erasure. □

7. The case of real ℓ_1^n

Theorem 4.5 gives that for every $N \geq n$, complex ℓ_1^n has a length N FUNTF. Lemma 4.3, used in the proof of Lemma 4.4, suggests that understanding FUNTFs in real ℓ_1^n will give insight into the situation for real spaces with a normalized 1-unconditional basis. The following are some partial results for the case of real ℓ_1^n, and by duality we have the same results for ℓ_∞^n as well.

Proposition 7.1. *For all $n \in \mathbb{N}$, ℓ_1^n has a FUNTF of length $n+1$.*

Proof. Note that we just need to consider the case $n \geq 3$ as every 2 dimensional Banach space with a symmetric basis has a FUNTF of all possible sizes. Let $(e_j)_{j=1}^n$ be the unit vector basis for ℓ_1^n and $(e_j^*)_{j=1}^n$ be the biorthogonal functionals. For $1 \leq j \leq n$ and $0 \leq a \leq 1$ we let

$$x_j = a e_j - \sum_{i \neq j} (1-a)(n-1)^{-1} e_i \quad \text{and} \quad x_{n+1} = \sum_{i=1}^n n^{-1} e_i.$$

The corresponding normalizing functionals are

$$x_j^* = e_j^* - \sum_{i \neq j} e_i^* \quad \text{and} \quad x_{n+1}^* = \sum_{i=1}^n e_i^*.$$

We will prove that there exists some constant $a \in (0,1)$ such that $(x_j, x_j^*)_{j=1}^{n+1}$ is a FUNTF. For each $1 \le j \le n+1$, it is clear that $\|x_j\| = \|x_j^*\| = x_j^*(x_j) = 1$. We now check the frame operator of $(x_j, x_j^*)_{j=1}^{n+1}$. For $1 \le m \le n$ we have that

$$\begin{aligned}\sum_{j=1}^{n+1} x_j^*(e_m)e_m^*(x_j) &= x_m^*(e_m)e_m^*(x_m) + \sum_{j\neq m,n+1} x_j^*(e_m)e_m^*(x_j) + x_{n+1}^*(e_m)e_m^*(x_{n+1}) \\ &= a + (n-1)(1-a)(n-1)^{-1} + n^{-1} = 1 + n^{-1}.\end{aligned}$$

For $1 \le m, k \le n$ with $m \neq k$ we have that

$$\begin{aligned}\sum_{j=1}^{n+1} x_j^*(e_m)e_k^*(x_j) &= x_m^*(e_m)e_k^*(x_m) + x_k^*(e_m)e_k^*(x_k) \\ &\quad + \sum_{j\neq k,m,n+1} x_j^*(e_m)e_k^*(x_j) + x_{n+1}^*(e_m)e_k^*(x_{n+1}) \\ &= -(1-a)(n-1)^{-1} - a + (n-2)(1-a)(n-1)^{-1} + n^{-1} \\ &= -a + (n-3)(1-a)(n-1)^{-1} + n^{-1}.\end{aligned}$$

This value is positive for $a = 0$ and negative for $a = 1$. Thus, there exists $a \in (0,1)$ such that $\sum_{j=1}^{n+1} x_j^*(e_m)e_k^*(x_j) = 0$. This proves that the frame operator of $(x_j, x_j^*)_{j=1}^{n+1}$ is $(1 + \frac{1}{n})$ times the identity. □

In the previous proposition, we gave a construction of a FUNTF of $n+1$ vectors in ℓ_1^n for $n \ge 3$. The proof only considered the case $n \ge 3$ because we already knew the result for $n = 2$. This is fortunate, because as the following proposition shows, the construction in Proposition 7.1 actually fails for $n = 2$.

Proposition 7.2. *Every FUNTF of odd length in ℓ_1^2 includes an element of the canonical basis (up to a sign).*

Proof. Suppose there is a FUNTF of length 3 in ℓ_1^2, say consisting of vectors x_1, x_2 and x_3, which does not include an element of the canonical basis or their negatives. This implies that all the coordinates of x_1, x_2 and x_3 are nonzero. Replacing x_i by $-x_i$ if necessary, we can assume that all three vectors have positive first coordinate. By reflecting the second coordinate, we can assume that both coordinates of x_1 are positive. Therefore, we may assume

$$x_1 = (a, 1-a), \quad x_2 = (b, \varepsilon_1(1-b)), \quad x_3 = (c, \varepsilon_2(1-c))$$

with $a, b, c \in (0,1)$ and $\varepsilon_1, \varepsilon_2 = \pm 1$. Their corresponding normalizing functionals must then be

$$x_1^* = (1,1), \quad x_2^* = (1,\varepsilon_1), \quad x_3^* = (1,\varepsilon_2).$$

Note that

$$x_1^* \otimes x_1 = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}, \quad x_2^* \otimes x_2 = \begin{pmatrix} b & \varepsilon_1(1-b) \\ \varepsilon_1 b & 1-b \end{pmatrix}, \quad \text{and}$$
$$x_3^* \otimes x_3 = \begin{pmatrix} c & \varepsilon_2(1-c) \\ \varepsilon_2 c & 1-c \end{pmatrix}.$$

It follows that in order to have a FUNTF, $a+b+c = 3/2$ and, moreover, we need to choose the signs $\varepsilon_1, \varepsilon_2$ in such a way that

$$a + \varepsilon_1 b + \varepsilon_2 c = 0 \qquad \text{and} \qquad 1 - a + \varepsilon_1(1-b) + \varepsilon_2(1-c) = 0.$$

However, adding both of these equations together makes $1 + \varepsilon_1 + \varepsilon_2 = 0$, impossible due to parity. This proves that there does not exist a length 3 FUNTF that does not contain one of the canonical basis vectors (up to a sign). Furthermore, the same parity argument shows that a FUNTF for ℓ_1^2 of any odd length must include an element of the canonical basis (up to a sign). □

The following proposition provides the examples we will need to answer the question of existence of FUNTFs in ℓ_1^n for $n = 3, 4$.

Proposition 7.3.

(a) *There exists a FUNTF of length* 5 *in* ℓ_1^3.
(b) *There exists a FUNTF of length* 6 *in* ℓ_1^4.
(c) *There exists a FUNTF of length* 7 *in* ℓ_1^4.

Proof. (a) We would like to use a collection of vectors

$$x_1 = (1,0,0), x_2 = (-a,b,b), x_3 = (-a,-b,b), x_4 = (-a,b,-b), x_5 = (-a,-b,-b),$$

with $a, b > 0$ and $a + 2b = 1$. (our choice of $-a$ instead of a is so that the frame looks more like a "pyramid" in ℓ_1^3). The corresponding normalizing functionals are

$$x_1^* = (1,0,0), x_2^* = (-1,1,1), x_3^* = (-1,-1,1), x_4^* = (-1,1,-1), x_5^* = (-1,-1,-1).$$

A calculation shows that

$$\sum_{j=1}^{5} x_j^* \otimes x_j = \begin{pmatrix} 1+4a & 0 & 0 \\ 0 & 4b & 0 \\ 0 & 0 & 4b \end{pmatrix},$$

so choosing $a = 1/6$ and $b = 5/12$ works.

(b) For $a, b, c, d > 0$ with $a + 2b = c + d = 1$, take

$$x_1 = (a, b, b, 0), x_2 = (a, -b, b, 0), x_3 = (a, b, -b, 0),$$
$$x_4 = (a, -b, -b, 0), x_5 = (c, 0, 0, d), x_6 = (c, 0, 0, -d),$$

and

$$x_1^* = (1, 1, 1, 0), x_2^* = (1, -1, 1, 0), x_3^* = (1, 1, -1, 0),$$
$$x_4^* = (1, -1, -1, 0), x_5^* = (1, 0, 0, 1), x_6^* = (1, 0, 0, -1).$$

A calculation shows that

$$\sum_{j=1}^{6} x_j^* \otimes x_j = \begin{pmatrix} 4a + 2c & 0 & 0 & 0 \\ 0 & 4b & 0 & 0 \\ 0 & 0 & 4b & 0 \\ 0 & 0 & 0 & 2d \end{pmatrix}.$$

We then choose $a = 1/4$, $b = 3/8$, $c = 1/4$, $d = 3/4$.

(c) The argument is quite similar: choose vectors as in the previous example with $a = 1/8$, $b = 7/16$, $c = 5/8$ and $d = 3/8$, together with

$$x_7 = (0, 0, 0, 1), \qquad x_7^* = (0, 0, 0, 1). \qquad \square$$

For general $n \in \mathbb{N}$, we know that ℓ_1^n has a FUNTF of length n and a FUNTF of length $n + 1$ (Propositions 4.1 and 7.1). As the union of FUNTFs is a FUNTF, in order to determine if ℓ_1^n has a FUNTF of all lengths at least the dimension we just need to find FUNTFs of lengths $n + 2, n + 3, ..., 2n - 1$. Thus, the previous proposition implies that we have a full positive answer in dimensions 2, 3, and 4.

Corollary 7.4. *ℓ_1^2, ℓ_1^3, and ℓ_1^4 all have FUNTFs of all lengths at least their dimension.*

8. Open problems

In the Hilbert space setting, every local minimizer of the frame potential is a global minimizer and hence a FUNTF [1]. As pointed out at the beginning of Section 4, the local minimizers of the frame potential are not understood in the general Banach space case.

Question 8.1. In a finite-dimensional Banach space, are the local minimizers of the frame potential FUNTFs? If not, what do they correspond to?

We showed in Section 4 that a large class of finite dimensional Banach spaces have finite unit norm tight frames of every length at least the dimension of the space. However, we do not have any examples where this is not possible.

Question 8.2. Does every n dimensional Banach space have a length N FUNTF for all $N \geq n$?

It seems very difficult to create a method of constructing FUNTFs of arbitrary size that works in any finite dimensional Banach space. Thus, it may be best to focus first on specific classical Banach spaces. We have shown in Section 4 that complex ℓ_1^n has a FUNTF of length N for all $n \leq N$ and in Section 7 that real ℓ_1^2, ℓ_1^3, and ℓ_1^4 each have FUNTFs of all lengths at least their dimension.

Question 8.3. Does real ℓ_1^n have a length N FUNTF for all $N \geq n$?

Note that for each n, there are only finitely many values of N for which we do not know the answer to the question above. Indeed, it follows from our results that for any $N \geq n(n-1)$ a FUNTF of length N does exist: if $N = n(n-1+m)+k$ with $0 \leq k \leq n-1$ and $m \geq 0$, we can take the union of k copies of a FUNTF of length $n+1$ and $n-1-k+m$ copies of a FUNTF of length n.

A finite dimensional Banach space X has a length N FUNTF if and only if a scalar multiple of the identity operator on X can be expressed as a sum of N normalized rank 1 projections. In the Hilbert space case, operators that can be written as sums of norm-one rank-one projections are characterized as positive operators with integer trace [28]. Operators on complex Banach spaces that can be written as sums of rank-one projections (dropping the norm one condition) can be characterized as well [2].

Question 8.4. Given a finite dimensional Banach space X, how can we characterize what operators may be expressed as sums of norm-one rank-one projections?

If X is not strictly convex then there exist rank-one operators on X where $\Pi_2(X, X)$ is not smooth, as X^* is isometric to a subspace of rank-one operators on $\Pi_2(X, X)$. However, we do not have any example of a finite dimensional Banach space X where $\Pi_2(X, X)$ is not smooth at an invertible operator. We used that $\Pi_2(X, X)$ is smooth at the identity to prove Theorem 2.4, which characterized FUNTFs in terms of the frame potential. Knowing that $\Pi_2(X, X)$ was smooth at some invertible operator T would be useful in studying approximate Schauder frames for X whose frame operator is normed by T.

Question 8.5. Let X be a finite dimensional Banach space. Is $\Pi_2(X, X)$ smooth at every invertible operator on X?

It is clear that for all $n > 1$, $\Pi_2(\ell_2^n, \ell_2^n)$ is smooth, and that $\Pi_2(\ell_1^n, \ell_1^n)$ and $\Pi_2(\ell_\infty^n, \ell_\infty^n)$ are not smooth. However, we do not have any results for other values of p.

Question 8.6. Let $n \in \mathbb{N}$ and $1 < p < \infty$. Is $\Pi_2(\ell_p^n, \ell_p^n)$ smooth?

In Proposition 6.1 we characterized the Schauder frames that minimize the reconstruction error due to the erasure of one coefficient as rescalings of FUNTFs. It would be quite interesting to say something about optimality when two frame coefficients are erased. In Hilbert spaces, the equiangular frames (when they exist) are optimal under two erasures. It would be quite interesting to find a generalization of equiangular to certain Banach spaces. This would require control of the norms of the operators

$$x_i^* \otimes x_i + x_j^* \otimes x_j, \qquad \text{for } i \neq j.$$

Question 8.7. What are some examples of $N \in \mathbb{N}$ and finite dimensional Banach spaces where we may characterize the length N Schauder frames that minimize the maximal error due to two erasures?

In Hilbert spaces, FUNTFs minimize the reconstruction error due to the loss of a single coefficient as well as minimize the mean squared error due to noise [20]. We have generalized this to all Banach spaces in the case of error due to the loss of a single coefficient, but we have not considered the error due to noise in a general Banach space.

Question 8.8. What are some examples of $N \in \mathbb{N}$ and finite dimensional Banach spaces where we may characterize the length N Schauder frames that minimize the mean squared error due to noise?

Declaration of Competing Interest

The authors have no competing interests for the research conducted in this paper.

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