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Operator p -compact mappings [☆]



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ABSTRACT

We introduce the classes of operator p -compact mappings and completely right p -nuclear operators, which are natural extensions to the operator space framework of their corresponding Banach operator ideals. We relate these two classes, define natural operator space structures and study several properties of these ideals. We show that the class of operator ∞ -compact mappings in fact coincides with a notion already introduced by Webster in the nineties (in a very different language). This allows us to provide an operator space structure to Webster's class.

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0. Introduction

Alexander Grothendieck, one of the most influential mathematicians of the 20th century, took his first steps in research in the area of functional analysis. In his early work, he developed a systematic theory of tensor products and norms [18], where he set the basis of what was later known as local theory — that is, the study of Banach spaces in terms of their finite-dimensional subspaces — and exhibited the importance of the use of tensor products in the theory of Banach (and locally convex) spaces. Grothendieck’s approach, although at first it received little notice given its complexity and the language in which it was written, was rediscovered at the end of the sixties by Lindenstrauss and Pełczyński [23] and inspired huge developments in Banach space theory. In particular, it inspired the creation of a whole new theory: the study of operator ideals (systematized by Pietsch and the German school [28,29]). In order to develop the metric theory of tensor products, Grothendieck gave a characterization of compactness in Banach spaces of independent interest [17, Chap. I, p. 112]: relatively compact sets are precisely those that lie within the absolutely convex hull of a null sequence. That is, a set K in a Banach space \mathbf{X} is relatively compact if and only if there is a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbf{X} such that

$$K \subseteq \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n| \leq 1 \right\} \text{ and } \lim_{n \rightarrow \infty} \|x_n\| = 0. \tag{1}$$

Inspired by Grothendieck’s result, Sinha and Karn [37] introduced the notion of relatively p -compact sets. This definition describes relatively p -compact sets as those which are determined in a way similar to (1), but by p -summable sequences instead of null ones. That is, if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, a subset $K \subset \mathbf{X}$ is called *relatively p -compact* if there exists a sequence $(x_n)_n$ in \mathbf{X} such that

$$K \subseteq \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n|^{p'} \leq 1 \right\} \text{ and } \sum_{n=1}^{\infty} \|x_n\|^p < \infty. \tag{2}$$

Of course, in the limiting case $p = 1$ we replace the definition above by $K \subseteq \{ \sum_{n=1}^{\infty} \alpha_n x_n : |\alpha_n| \leq 1 \}$ and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Thus, classical compact sets can then be seen as “ ∞ -compact”. Moreover, the following monotonicity relation holds: if $1 \leq q \leq p \leq \infty$, any relatively q -compact set is in fact relatively p -compact. Therefore, p -compactness reveals “finer and subtle” structures on compact sets.

Having this definition at hand, it is natural to think about extending the notion of compactness to linear mappings. By simple analogy, p -compact mappings arise as those which map the unit ball into a relatively p -compact set. This definition, which comes from Sinha and Karn’s original paper [37], turns out to yield a normed operator ideal, denoted by \mathcal{K}_p (see the precise definition in Section 3). In recent years there has been great interest in this class — and in some notions that naturally relate to it — from a variety of perspectives including the theory of operator ideals, the theory of tensor

norms, the interplay with various approximation properties, structural properties of sets and sequences and infinite dimensional complex analysis (see, e.g. [9,7,26,16,22,31,24,1,14] and the references therein).

The main goal of the present paper is to initiate the study of a noncommutative version of p -compact mappings, specifically in the context of operator spaces. Recall that an operator space is a Banach space \mathbf{X} with an extra “matricial norm structure”: in addition to the norm on \mathbf{X} , we have norms on all the spaces $M_n(\mathbf{X})$ of $n \times n$ matrices with entries from \mathbf{X} (and these norms must satisfy certain consistency requirements). The morphisms are no longer just bounded maps, but the *completely bounded ones*: they are required to be uniformly bounded on all the matricial levels. Operator spaces are thus a quantized or noncommutative version of Banach spaces, giving rise to a theory that not only is mathematically attractive but it is also naturally well-positioned to have applications to quantum physics. The systematic study of operator spaces started with Ruan’s thesis, and was developed mainly by Effros and Ruan, Blecher and Paulsen, Pisier and Junge (see the monographs [20,33,3,12,34] and the references therein).

Due to their very definition, it is natural to investigate to what extent the classical theory of Banach spaces can be translated to the noncommutative context of operator spaces. Though some properties do carry over, many do not and these differences are one of the reasons making the new theory so interesting. In particular, both ideals of operators and tensor norms have inspired important developments in the operator space setting. Noncommutative versions of nuclear, integral, summing, and other ideals of operators have played significant roles in, e.g., [12,33,19,13,21], whereas in addition to the usefulness of various specific tensor norms for operator spaces (most notably the Haagerup one), Blecher and Paulsen [4] have showed that the elementary theory of tensor norms of Banach spaces carries over to operator spaces, initiating a “tensor norm program” for operator spaces further developed in [2]. However, the abstract tool of associating ideals of operators and tensor norms in the sense of [8, Chap. 17] does not appear to have been developed yet. In the present paper we take some small steps in this direction, which we need throughout our definition and study of the new class of *operator p -compact mappings* (which we denote by \mathcal{K}_p^o). A much more thorough study of the relationships between tensor norms and ideals of mappings in the operator space category will appear in [6].

One way to define this class is to rely on certain factorizations that characterize p -compact mappings via right p -nuclear maps. To this end, we study and define the ideal of *completely right p -nuclear mappings*, \mathcal{N}_p^o (which are, in some way, a transposed version of the class given in [19, Def. 3.1.3.1.] called p -nuclear operators). In particular, \mathcal{K}_p^o is the surjective hull of \mathcal{N}_p^o , as shown by Delgado, Piñeiro and Serrano [9, Prop. 3.11] and by Pietsch [31, Thm. 1] in the Banach space context. This relation among these two classes enables us to give an appropriate operator space structure to \mathcal{K}_p^o .

At the end of the nineties, Webster (a student of Effros) proposed in his doctoral thesis [38] several ways to extend the notion of compactness to the operator space framework (see also [39]). One of these notions, which he called *operator compactness*, is in effect a

noncommutative version of Grothendieck’s characterization (being in the absolute convex hull of a null sequence as in Equation (1)). This also gave raise to the notion of *operator compact mappings*. Therefore, another path to translate the concept of p -compactness to the category of operator spaces is based on Webster’s ideas. We show that these two possible ways to define p -compactness in the operator space setting (the one which involves the factorization through completely right p -nuclear mappings and the one based on Webster’s work) in fact coincide. This allows us to endow Webster’s class (of operator compact mappings) with an operator space structure.

The article is organized as follows. In Section 1 we present the notation and some basic concepts in the theory of operator spaces. In Section 2 we introduce the notion of completely right p -nuclear mappings, first in tensor terms and then proving a characterization in terms of factorizations. It should be mentioned that for $p = 1$, this concept coincides with the completely nuclear mappings already studied in [12, Sec. 12.2]. In Section 3 we introduce a notion of operator p -compact mappings based on factorization schemes, and give an operator space structure to this class by relating it to the aforementioned right p -nuclear mappings. Additionally, we characterize operator p -compact mappings in terms of a noncommutative notion of p -compact sets. We end the article with some open questions.

1. Preliminaries

Our Banach space notation is quite standard, and follows closely that of [8,36]. The injective tensor product of Banach spaces will be denoted by \otimes_ε . For a Banach space \mathbf{X} and $1 \leq p < \infty$, we denote by $\ell_p(\mathbf{X})$ and $\ell_p^w(\mathbf{X})$ the spaces of p -summable and weakly p -summable sequences in \mathbf{X} , respectively, with their usual norms

$$\|(x_n)\|_{\ell_p(\mathbf{X})} = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}, \quad \|(x_n)\|_{\ell_p^w(\mathbf{X})} = \sup_{\|x'\| \leq 1} \left(\sum_{n=1}^{\infty} |x'(x_n)|^p \right)^{1/p}$$

where x' is taken in \mathbf{X}' , the dual of \mathbf{X} . The obvious modifications are adapted to the case $p = \infty$. The unit ball of \mathbf{X} is denoted by $B_{\mathbf{X}}$.

We only assume familiarity with the basic theory of operator spaces; the books [34] and [12] are excellent references. Our notation follows closely that from [33,34], with one main exception: we denote the dual of a space E by E' .

Throughout the article E and F will be operator spaces. For each n , $M_n(E)$ will denote the space of $n \times n$ matrices of elements of E . We will just write M_n when E is the scalar field. The spaces $M_{n \times m}(E)$ and $M_{n \times m}$ are defined analogously. For a linear mapping $T : E \rightarrow F$, we will denote its n -amplification by $T_n : M_n(E) \rightarrow M_n(F)$. The space of completely bounded linear mappings from E into F will be noted by $\mathcal{CB}(E, F)$. The corresponding bilinear notion is the following: a bilinear map $T : E_1 \times E_2 \rightarrow F$ is jointly completely bounded if there exists a constant $C \geq 0$ such that for any matrices $(x_{ij})_{i,j=1}^n \in M_n(E_1)$ and $(y_{k\ell})_{k,\ell=1}^m \in M_m(E_2)$ we have

$$\|(T(x_{ij}, y_{k\ell}))\|_{M_{nm}(F)} \leq \|(x_{ij})_{i,j=1}^n\|_{M_n(E_1)} \|(y_{k\ell})_{k,\ell=1}^m\|_{M_m(E_2)}.$$

The least such C is denoted $\|T\|_{\text{jcb}}$. As in [12, Sec. 10.1] we will denote by $M_\infty(E)$ the (operator) space of all infinite matrices (x_{ij}) with coefficients in E such that the truncated matrices are uniformly bounded, i.e., $\sup_{n \in \mathbb{N}} \|(x_{ij})_{i,j=1}^n\|_{M_n(E)} < \infty$. This supremum corresponds to the norm of the element (x_{ij}) in $M_\infty(E)$. The space M_∞ of bounded scalar infinite matrices is naturally identified with $\mathcal{B}(\ell_2)$; more generally, if E is concretely realized as a subspace of $B(H)$, then $M_\infty(E)$ is realized as a subspace of $B(\ell_2 \otimes_2 H)$ where \otimes_2 denotes the Hilbertian tensor product (the latter is the approach taken in [33]).

Our notation for the minimal and the projective operator space tensor products will be, respectively, \otimes_{\min} and \otimes_{proj} . By an operator space cross norm α we mean an assignment of an operator space $E \otimes_\alpha F$ to each pair (E, F) of operator spaces, in such a way that $E \otimes_\alpha F$ is the algebraic tensor product $E \otimes F$ together with a matricial norm structure on $E \otimes F$ that we write as α_n or $\|\cdot\|_{\alpha_n}$, and such that $\alpha_{nm}(x \otimes y) = \|x\|_{M_n(E)} \cdot \|y\|_{M_m(F)}$ for every $x \in M_n(E)$, $y \in M_m(F)$. This implies that the identity map $E \otimes_{\text{proj}} F \rightarrow E \otimes_\alpha F$ is completely contractive [4, Thm. 5.5]. If in addition the identity map $E \otimes_\alpha F \rightarrow E \otimes_{\min} F$ is also completely contractive, we say that α is an *operator space tensor norm*. Moreover, an operator space tensor norm α is called *uniform* if additionally for any operator spaces E_1, E_2, F_1, F_2 , the map $\otimes_\alpha : \mathcal{CB}(E_1, E_2) \times \mathcal{CB}(F_1, F_2) \rightarrow \mathcal{CB}(E_1 \otimes_\alpha E_2, F_1 \otimes_\alpha F_2)$ given by $(S, T) \mapsto S \otimes_\alpha T$ is jointly completely contractive.

If α is an operator space tensor norm, the completion of the tensor product $E \otimes_\alpha F$ will be denoted by $E \widehat{\otimes}_\alpha F$. A degree of caution is required when consulting different works dealing with operator space tensor products, since the term “tensor norm” is not always taken to have the exact same meaning. We point out that the definition we use was introduced in [4] and it is slightly different than those of [12, 11], though not in any significant way. An operator space tensor norm α is said to be *finitely generated* if for any operator spaces E and F and $u \in M_n(E \otimes F)$, $\alpha_n(u; E, F) = \inf \{ \alpha_n(u; E_0, F_0) : u \in M_n(E_0 \otimes F_0) \}$, where the infimum is taken over finite-dimensional subspaces $E_0 \subset E$ and $F_0 \subset F$. For operator space tensor norms α and β and a constant c , we write “ $\alpha \leq c\beta$ on $E \otimes F$ ” to indicate that the identity map $E \otimes_\beta F \rightarrow E \otimes_\alpha F$ has cb-norm at most c . If no reference to spaces is made, we mean that the inequality holds for any pair of operator spaces. A linear map $Q : E \rightarrow F$ between operator spaces is called a complete 1-quotient if it is onto and the associated map from $E/\ker(Q)$ to F is a completely isometric isomorphism. These maps are called complete metric surjections in [34, Sec. 2.4], where it is proved that a linear map $u : E \rightarrow F$ is a complete 1-quotient if and only if its adjoint $u' : F' \rightarrow E'$ is a completely isometric embedding.

As can already be seen in the introduction, the ℓ_p spaces play a fundamental role in defining and studying the notion of p -compactness in Banach spaces. The noncommutative analog of ℓ_p is the Schatten class S_p , which is defined for $1 \leq p < \infty$ as the space of all compact operators T on ℓ_2 such that $\text{tr}|T|^p < \infty$ equipped with the norm

$\|T\|_{S_p} = (\operatorname{tr} |T|^p)^{1/p}$; in the case $p = \infty$, we denote by S_∞ the space of all compact operators on ℓ_2 endowed with the operator norm. Of course the reason for this analogy is that ℓ_p can be identified with the diagonal operators in S_p , and the commutative/noncommutative distinction comes from the fact that any two diagonal operators commute. However, to fully realize S_p as the noncommutative analog of ℓ_p we additionally need an operator space structure on S_p . The case of S_∞ poses no issue, as any C^* -algebra has a canonical operator space structure [12, p. 21]; using the duality $S'_1 = B(\ell_2)$ we can also get a natural operator space structure on S_1 . Pisier’s theory of complex interpolation for operator spaces [32, Sec. 2], [34, Sec. 2.7] then allows us to give an operator space structure to each of the intermediate S_p spaces; as remarked in [34, p. 141], this abstract approach realizes S_p as a subspace of a $B(H)$ space in a highly nonstandard way. More generally, following [33], for an operator space E and $1 \leq p \leq \infty$ we can define an operator space which is an E -valued version of S_p and will be denoted by $S_p[E]$: $S_\infty[E]$ is the minimal operator space tensor product of S_∞ and E , $S_1[E]$ as the operator space projective tensor product of S_1 and E , and once again in the case $1 < p < \infty$ we define $S_p[E]$ via complex interpolation between $S_\infty[E]$ and $S_1[E]$. For $1 < p \leq \infty$, the dual of $S_p[E]$ can be canonically identified with $S_{p'}[E']$, where p' satisfies $1/p + 1/p' = 1$ [33, Cor. 1.8]. In the discussion above, if we replace S_1 by the space S_1^n of $n \times n$ matrices with the trace norm, and S_∞ by the space M_n , we can analogously construct operator spaces S_p^n and $S_p^n[E]$.

Finally, a remark about our use of the word *operator*. In the Banach space literature it is usual to speak of p -compact *operators* rather than p -compact *mappings*, and similarly for other various classes of bounded linear transformations between normed spaces. In order to avoid confusions, throughout this paper we reserve the word *operator* for notions that are noncommutative in nature (as in *operator space*), and use the word *mapping* to refer to bounded linear transformations.

2. Completely right p -nuclear mappings

In this section we introduce the notion of completely right p -nuclear mappings, inspired by the definitions in the Banach space setting. For $p = 1$, this concept coincides with the completely nuclear mappings studied in [12, Sec. 12.2]. Recall that in the Banach space setting a linear mapping $T : \mathbf{X} \rightarrow \mathbf{Y}$ is *right p -nuclear* ([35], [36, Sec. 6.2]) if T can be written as

$$T = \sum_{n=1}^{\infty} x'_n \otimes y_n,$$

where $(x'_n) \in \ell_{p'}^w(\mathbf{X}')$, $(y_n) \in \ell_p(\mathbf{Y})$. Moreover, the right p -nuclear norm of T is defined as

$$\nu^p(T) := \inf \{ \| (x'_n) \|_{\ell_{p'}^w(\mathbf{X}')} \cdot \| (y_n) \|_{\ell_p(\mathbf{Y})} \},$$

where the infimum is taken all over possible representations of T as above. The class of all right p -nuclear mappings is denoted by \mathcal{N}^p . This definition is known to be equivalent to having a factorization

$$\begin{array}{ccc}
 \mathbf{X} & \xrightarrow{T} & \mathbf{Y} \\
 U \downarrow & & \uparrow V \\
 \ell_{p'} & \xrightarrow{D_\lambda} & \ell_1,
 \end{array} \tag{3}$$

where U and V are bounded mappings, $\lambda \in \ell_p$, and D_λ stands for the diagonal multiplication mapping $(x_n) \mapsto (\lambda_n x_n)$. Indeed, if $T = VD_\lambda U$ then defining $x'_n := e'_n \circ U$ and $y_n := Ve_n$ we have that $T = \sum_{n=1}^\infty x'_n \otimes y_n$, where (e'_n) and (e_n) are the canonical unit vectors of ℓ_p and ℓ_1 respectively. Reciprocally, if T is written as $\sum_{n=1}^\infty x'_n \otimes y_n$ then defining $U : \mathbf{X} \rightarrow \ell_{p'}$ by $U(x) := (x'_n(x))$ and $V : \ell_1 \rightarrow \mathbf{Y}$ by $V((a_n)) = \sum_{n=1}^\infty a_n y_n$, we have a factorization as in (3). Moreover, $\nu^p(T) = \inf\{\|U\| \|D_\lambda\| \|V\|\}$ where the infimum runs over all factorizations as above.

Recall that, given Banach spaces \mathbf{X} and \mathbf{Y} and $1 \leq p \leq \infty$, the p -right Chevet-Saphar tensor norm ([36, Chap. 6], [8, Chap. 12]) d_p of a tensor $u \in \mathbf{X} \otimes \mathbf{Y}$ is defined by

$$d_p(u) = \inf \left\{ \|(x_j)\|_{\ell_{p'} \otimes_\varepsilon X} \|(y_j)\|_{\ell_p(Y)} : u = \sum_j x_j \otimes y_j \right\}, \tag{4}$$

where the infimum runs over all the possible ways in which the tensor u can be written as a finite sum as above.

From the definition it follows that right p -nuclear mappings between the Banach spaces \mathbf{X} and \mathbf{Y} are exactly those mappings which are in the range of the canonical inclusion $\mathbf{X}' \widehat{\otimes}_{d_p} \mathbf{Y} \rightarrow \mathbf{X}' \widehat{\otimes}_\varepsilon \mathbf{Y}$, and the right p -nuclear norm coincides with the quotient norm inherited from this inclusion (see also the analogous results given in [8, 22.3], [36, Sec. 6.2] and [10, Prop. 5.23]).

Moving to the operator space realm, for $1 \leq p \leq \infty$, in [5] the p -right Chevet-Saphar operator space tensor norm d_p^o is defined in the following way: given operator spaces E and F , for $u \in E \otimes F$,

$$d_p^o(u) = \inf \left\{ \|(x_{ij})\|_{S_{p'}^n \otimes_{\min} E} \|(y_{ij})\|_{S_p^n[F]} : u = \sum_{i,j=1}^n x_{ij} \otimes y_{ij} \right\}, \tag{5}$$

where the infimum runs over all the possible ways in which the tensor u can be written as a finite sum as above.

The operator space structure of the tensor $E \widehat{\otimes}_{d_p^o} F$ is given by the following 1-quotient (see [5, Sec. 3])

$$Q^p : (S_{p'} \widehat{\otimes}_{\min} E) \widehat{\otimes}_{\text{proj}} S_p[F] \rightarrow E \widehat{\otimes}_{d_p^o} F,$$

where $\widehat{\otimes}$ means the completion of the corresponding tensor product.

Motivated by the definition given in the Banach setting, we introduce a corresponding notion of p -nuclear mappings in the category of operator spaces.

Definition 2.1. Let $1 \leq p \leq \infty$. We say that a linear mapping $T : E \rightarrow F$ between operator spaces E and F is *completely right p -nuclear* if it corresponds to an element in the range of the canonical inclusion

$$J^p : E' \widehat{\otimes}_{d_p^o} F \rightarrow E' \widehat{\otimes}_{\min} F.$$

We denote the space of all such mappings by $\mathcal{N}_o^p(E, F)$, and we endow it with the quotient structure $(E' \widehat{\otimes}_{d_p^o} F) / \ker J^p$. In particular, its norm – that we denote by ν_o^p – is the quotient norm.

For future reference, it is important to remark that since the operator space structure on $E' \widehat{\otimes}_{d_p^o} F$ is itself coming from a quotient, the above definition is equivalent to being in the range of

$$J^p \circ Q^p : (S_{p'} \widehat{\otimes}_{\min} E') \widehat{\otimes}_{\text{proj}} S_p[F] \rightarrow E' \widehat{\otimes}_{\min} F,$$

and this still induces the same quotient structure.

Following [12, Sec. 12.2], a *mapping ideal* $(\mathfrak{A}, \mathbf{A})$ is an assignment, for each pair of operator spaces E, F , of a linear space $\mathfrak{A}(E, F) \subseteq \mathcal{CB}(E, F)$ together with an operator space matrix norm \mathbf{A} on $\mathfrak{A}(E, F)$ such that

- (i) The identity map $\mathfrak{A}(E, F) \rightarrow \mathcal{CB}(E, F)$ is a complete contraction.
- (ii) The ideal property: whenever $T \in M_n(\mathfrak{A}(E, F))$, $r \in \mathcal{CB}(E_0, E)$ and $s \in \mathcal{CB}(F, F_0)$, it follows that $\mathbf{A}_n(s_n \circ T \circ r) \leq \|s\|_{\text{cb}} \mathbf{A}_n(T) \|r\|_{\text{cb}}$.

Let us start by proving that \mathcal{N}_o^p is a mapping ideal. We observe, also, that each space $\mathcal{N}_o^p(E, F)$ is complete.

Proposition 2.2. \mathcal{N}_o^p is a mapping ideal, for $1 \leq p \leq \infty$.

Proof. Given $T \in M_n(\mathcal{N}_o^p(E; F))$, the definition of right p -nuclearity yields $\|T\|_{M_n(\mathcal{CB}(E; F))} \leq \|T\|_{M_n(\mathcal{N}_o^p(E; F))}$. Also for $S \in \mathcal{CB}(E_0, E)$ and $R \in \mathcal{CB}(F, F_0)$ we have $R_n T S \in M_n(\mathcal{N}_o^p(E_0, F_0))$ and

$$\|R_n T S\|_{M_n(\mathcal{N}_o^p(E_0, F_0))} \leq \|R\|_{\text{cb}} \|T\|_{M_n(\mathcal{N}_o^p(E; F))} \|S\|_{\text{cb}}.$$

Indeed, by definition there exists $t \in M_n(E' \widehat{\otimes}_{d_p^o} F)$ such that $(J^p)_n(t) = T$. Since d_p^o is a uniform operator space tensor norm,

$$d_p^o((S' \otimes R)_n(t)) \leq \|S'\|_{cb} d_p^o(t) \|R\|_{cb} = \|S\|_{cb} d_p^o(t) \|R\|_{cb}.$$

Note that $(J^p)_n((S' \otimes R)_n(t)) = R_n T S$, so $R_n T S \in M_n(\mathcal{N}_o^p(E_0, F_0))$ and

$$\|R_n T S\|_{M_n(\mathcal{N}_o^p(E_0, F_0))} \leq \|S\|_{cb} d_p^o(t) \|R\|_{cb}.$$

Taking the infimum over all t we get the desired conclusion. \square

The next result shows that the formula given in Equation (5) can be extended for tensors that lie in the completion $E \widehat{\otimes}_{d_p^o} F$. To that end, we will want to consider the elements of the spaces $S_p \otimes_{\min} E$ and $S_p[E]$ as infinite matrices with entries in E . In the first case the meaning is clear: since $S_p \otimes_{\min} E$ completely isometrically embeds into $\mathcal{CB}(S'_p, E)$, we identify $X \in S_p \otimes_{\min} E$ with the infinite E -valued matrix that arises from applying X (considered as a map $S'_p \rightarrow E$) to the matrix units in S'_p . For $Y \in S_p[E]$ the identification as an infinite matrix is not immediately clear, since $S_p[E]$ was constructed using complex interpolation. The reader is invited to check [33, p. 18–20] for the full details, but in a nutshell the idea is as follows. If R and C are the row and column Hilbert spaces, respectively, and \otimes_h is the Haagerup tensor product, it is known that there are identifications $S_1[E] = R \otimes_h E \otimes_h C$ and $S_\infty[E] = C \otimes_h E \otimes_h R$. This yields the identifications as infinite matrices with values in E for the elements of the spaces $S_1[E]$ and $S_\infty[E]$, and these identifications are compatible with the interpolation procedure that generates the rest of the $S_p[E]$ spaces: [33, Thm. 1.1] states that $S_p[E]$ can be canonically identified with a space of the form $R(1 - \theta) \otimes_h E \otimes_h R(\theta)$, where $R(\theta) = (R, C)_\theta$. When thinking of the elements of $S_p[E]$ as infinite matrices in this way, an important feature of the identification is the fact that $(x_{ij})_{i,j=1}^\infty \in S_p[E]$ is the limit (in $S_p[E]$) of its truncations $(x_{ij})_{i,j=1}^n$ [33, Lemma 1.12] (of course, we think of the truncations as infinite matrices by padding them with zeros). Let us now consider $(x_{ij})_{i,j=1}^\infty \in S_{p'} \widehat{\otimes}_{\min} E$ and $(y_{ij})_{i,j=1}^\infty \in S_p[F]$, where E and F are operator spaces. From the definition of Q^p , it is clear that $Q^p((x_{ij})_{i,j=1}^\infty \otimes (y_{ij})_{i,j=1}^n) = \sum_{i,j=1}^n x_{ij} \otimes y_{ij}$. Since $(y_{ij})_{i,j=1}^n$ converges to $(y_{ij})_{i,j=1}^\infty$ in $S_p[F]$, by continuity of Q^p we get $Q^p((x_{ij})_{i,j=1}^\infty \otimes (y_{ij})_{i,j=1}^\infty) = \lim_{n \rightarrow \infty} \sum_{i,j=1}^n x_{ij} \otimes y_{ij}$. For simplicity, this limit will sometimes be denoted by $\sum_{i,j} x_{ij} \otimes y_{ij}$.

Theorem 2.3. *Let $1 \leq p \leq \infty$, let E and F be operator spaces, and $u \in E \widehat{\otimes}_{d_p^o} F$. Then*

$$\|u\|_{d_p^o} = \inf \left\{ \|(x_{ij})\|_{S_{p'} \widehat{\otimes}_{\min} E} \|(y_{ij})\|_{S_p[F]} : u = Q^p((x_{ij}) \otimes (y_{ij})) = \sum_{i,j} x_{ij} \otimes y_{ij} \right\}.$$

Proof. Let $u \in E \widehat{\otimes}_{d_p^o} F$. It is clear that $\|u\|_{d_p^o} \leq \|(x_{ij})\|_{S_{p'} \widehat{\otimes}_{\min} E} \|(y_{ij})\|_{S_p[F]}$, for any representation of $u = Q^p((x_{ij}) \otimes (y_{ij}))$, since Q^p is a complete contraction.

Also, for every $\eta > 0$ there is a sequence $(u_m) \in E \otimes F$ such that $u = \sum_m u_m \in E \widehat{\otimes}_{d_p^o} F$ with $d_p^o(u_1) < d_p^o(u) + \eta$ and $d_p^o(u_m) \leq \frac{\eta^2}{4^m}$ for every $m \geq 2$.

Now, we can write $u_1 := \sum_{i,j=1}^{k_1} x_{ij} \otimes y_{ij}$ where $\left\| (x_{ij})_{i,j=1}^{k_1} \right\|_{S_p^{k_1} \otimes_{\min} E} \leq d_p^o(u) + \eta$ and $\left\| (y_{ij})_{i,j=1}^{k_1} \right\|_{S_p[F]} \leq 1$. Also, for $m \geq 2$, we may represent $u_m := \sum_{i,j=k_{m-1}+1}^{k_m} x_{ij} \otimes y_{ij}$ with $\left\| (x_{ij})_{i,j=k_{m-1}+1}^{k_m} \right\|_{S_{p'}^{k_m} \otimes_{\min} E} \leq \frac{\eta}{2^m}$ and $\left\| (y_{ij})_{i,j=k_{m-1}+1}^{k_m} \right\|_{S_p[F]} \leq \frac{\eta}{2^m}$. Padding with zeros, we construct infinite matrices (x_{ij}) and (y_{ij}) . By the triangle inequality we derive $\left\| (y_{ij}) \right\|_{S_p[F]} \leq 1 + \eta \sum_{m=2}^{\infty} 2^{-m}$ and also $\left\| (x_{ij}) \right\|_{S_{p'} \widehat{\otimes}_{\min} E} \leq d_p^o(u) + \eta + \eta \sum_{m=2}^{\infty} 2^{-m}$. To conclude the proof we just need to observe that $Q^p((x_{ij}) \otimes (y_{ij})) = u$, which follows from $Q^p((x_{ij}) \otimes (y_{ij})_{i,j=1}^{k_n}) = \sum_{m=1}^n u_m$ and the fact that $(y_{ij})_{i,j=1}^{k_n}$ converges to (y_{ij}) in $S_p[F]$ as $n \rightarrow \infty$. \square

We now introduce a non-commutative version of the sequence space $\ell_p^w(E)$, namely

$$S_p^w[E] := \{x \in M_{\infty}(E) : \sup_N \left\| (x_{ij})_{i,j=1}^N \right\|_{S_p^N \widehat{\otimes}_{\min} E} < \infty\}.$$

It can be easily seen that this is an operator space endowed with the matricial norm structure given by

$$\left\| ((x_{ij}^{kl})_{i,j}^n)_{k,l=1}^n \right\|_{M_n(S_p^w[E])} := \sup_N \left\| ((x_{ij}^{kl})_{i,j=1}^N)_{k,l=1}^n \right\|_{M_n(S_p^N \widehat{\otimes}_{\min} E)}.$$

Recall that $\ell_p^w(E)$ can be identified with the space of bounded linear mappings from $\ell_{p'}$ to E [8, Prop. 8.2.(1)]. The following is the analogous statement for $S_p^w[E]$.

Lemma 2.4. *For $1 \leq p \leq \infty$, we have the following completely isometric identification*

$$S_p^w[E] = \mathcal{CB}(S_{p'}, E).$$

Proof. Let us see first that the spaces $S_p^w[E]$ and $\mathcal{CB}(S_{p'}, E)$ are isometrically isomorphic. For each $N \in \mathbb{N}$, we denote $\rho_N : M_{\infty}(E) \rightarrow M_N(E)$ the N -truncation given by $\rho_N((x_{ij})_{ij}) = (x_{ij})_{i,j=1}^N$. For each $X \in S_p^w[E]$,

$$\|X\|_{S_p^w[E]} = \sup_N \|\rho_N(X)\|_{S_p^N \widehat{\otimes}_{\min} E} = \sup_N \|\rho_N(X)\|_{\mathcal{CB}(S_{p'}^N, E)}$$

since the spaces $S_p^N \widehat{\otimes}_{\min} E$ and $\mathcal{CB}(S_{p'}^N, E)$ are isometric. Thus, the mapping $T_X : \bigcup_{N=1}^{\infty} S_{p'}^N \rightarrow E$ is well defined (given $A \in S_{p'}^N$ for some $N \in \mathbb{N}$, $T_X(A) := \rho_N(X)(A)$) and uniquely extends to $S_{p'}$ by density. The same holds with the n -amplification $(T_X)_n : M_n(\bigcup_{N=1}^{\infty} S_{p'}^N) \rightarrow M_n(E)$ (whose norm is obviously bounded by $\|X\|_{S_p^w[E]}$), and extends to $M_n(S_{p'})$. Hence, $T_X \in \mathcal{CB}(S_{p'}, E)$ with $\|T_X\|_{\mathcal{CB}(S_{p'}, E)} \leq \|X\|_{S_p^w[E]}$.

Conversely, for $T \in \mathcal{CB}(S_{p'}, E)$, denoting by $i_N : S_{p'}^N \rightarrow S_{p'}$ the canonical completely isometric inclusion, we derive that $T \circ i_N \in \mathcal{CB}(S_{p'}^N, E)$ with $\|T \circ i_N\|_{\mathcal{CB}(S_{p'}^N, E)} \leq \|T\|_{\mathcal{CB}(S_{p'}, E)}$, for each $N \in \mathbb{N}$. Now, defining $X_T \in M_{\infty}(E)$ by $X_T = (Te_{ij})_{i,j}$ it is plain, through the identity $S_p^N \widehat{\otimes}_{\min} E = \mathcal{CB}(S_{p'}^N, E)$, that

$$\|X_T\|_{S_p^w[E]} = \sup_N \|\rho_N(X_T)\|_{S_p^N \widehat{\otimes}_{\min} E} = \sup_N \|T \circ i_N\|_{\mathcal{CB}(S_p^N, E)} \leq \|T\|_{\mathcal{CB}(S_{p'}, E)}.$$

Hence, we have proved the isometry $S_p^w[E] = \mathcal{CB}(S_{p'}, E)$. To see that this isometry is complete, we need to show that, for each n , the spaces $M_n(S_p^w[E])$ and $M_n(\mathcal{CB}(S_{p'}, E))$ are isometric. Recall that

$$M_n(\mathcal{CB}(S_{p'}, E)) \stackrel{\cong}{=} \mathcal{CB}(S_{p'}, M_n(E)) \stackrel{\cong}{=} S_p^w[M_n(E)],$$

where the last equality follows from the first part of the proof. To finish it only remains to see that $S_p^w[M_n(E)] \stackrel{\cong}{=} M_n(S_p^w[E])$. Indeed, by [12, Cor. 8.1.3 and Cor. 8.1.7], for all N the spaces $S_p^N \widehat{\otimes}_{\min} M_n(E)$ and $M_n(S_p^N \widehat{\otimes}_{\min} E)$ are completely isometric. Therefore, for each $X \in M_n(S_p^w[E])$, it holds

$$\begin{aligned} \|X\|_{M_n(S_p^w[E])} &= \sup_N \|(\rho_N)_n(X)\|_{M_n(S_p^N \widehat{\otimes}_{\min} E)} = \sup_N \|(\rho_N)_n(X)\|_{S_p^N \widehat{\otimes}_{\min} M_n(E)} \\ &= \|X\|_{S_p^w[M_n(E)]}. \quad \square \end{aligned}$$

As in the Banach space setting, we can replace $S_{p'} \widehat{\otimes}_{\min} E$ by $S_{p'}^w[E]$ in the quotient description of the norm d_p^o . Indeed, let

$$\widetilde{Q}^p : S_{p'}^w[E] \widehat{\otimes}_{\text{proj}} S_p[F] \rightarrow E \widehat{\otimes}_{d_p^o} F$$

be the canonical mapping given by $\widetilde{Q}^p((x_{ij}) \otimes (y_{ij})) = \sum_{i,j} x_{ij} \otimes y_{ij}$, for $(x_{ij}) \in S_{p'}^w[E]$ and $(y_{ij}) \in S_p[F]$. To see that this is well defined, denote by $\tau^k : S_{p'}^w[E] \rightarrow S_{p'}^k \otimes_{\min} E$ the truncation mapping given by $\tau^k((x_{ij})_{i,j}) = (x_{ij})_{i,j=1}^k$. It is plain, due to the operator space structure of $S_{p'}^w[E]$, that τ^k is a complete contraction. We note by $\widetilde{\tau}^k$ the “same” truncation mapping (that is, given by the same formula as τ^k) but on a different space, $\widetilde{\tau}^k : S_p[F] \rightarrow S_p[F]$.

Let $(x_{ij}) \in S_{p'}^w[E]$ and $(y_{ij}) \in S_p[F]$. To see that $\sum_{i,j} x_{ij} \otimes y_{ij}$ converges in $E \widehat{\otimes}_{d_p^o} F$ we prove that (u_k) is a Cauchy sequence, where $u_k = \sum_{i,j=1}^k x_{ij} \otimes y_{ij}$. By definition, if $k \geq l$,

$$d_p^o(u_k - u_l) \leq \|\tau^k(x_{ij}) - \tau^l(x_{ij})\|_{S_{p'}^k \otimes_{\min} E} \|\widetilde{\tau}^k(y_{ij}) - \widetilde{\tau}^l(y_{ij})\|_{S_p[F]}.$$

Note that $\|\tau^k(x_{ij}) - \tau^l(x_{ij})\|_{S_{p'}^k \otimes_{\min} E} \leq 2\|(x_{ij})\|_{S_{p'}^w[E]}$ and $\|\widetilde{\tau}^k(y_{ij}) - \widetilde{\tau}^l(y_{ij})\|_{S_p[F]} \leq \varepsilon$ for k, l sufficiently large, by [33, Lemma 1.12]. So, $(u_k)_k$ converges to $u = \sum_{i,j} x_{ij} \otimes y_{ij}$ and $d_p^o(u) \leq \|(x_{ij})\|_{S_{p'}^w[E]} \|(y_{ij})\|_{S_p[F]}$. Thus, \widetilde{Q}^p is well defined.

Theorem 2.5. *For $1 \leq p \leq \infty$, the mapping $\widetilde{Q}^p : S_{p'}^w[E] \widehat{\otimes}_{\text{proj}} S_p[F] \rightarrow E \widehat{\otimes}_{d_p^o} F$ is a complete 1-quotient.*

Proof. We have just to prove that \tilde{Q}^p is a complete contraction: since Q^p is a complete 1-quotient and $S_{p'} \widehat{\otimes}_{\min} E \hookrightarrow S_{p'}^w[E]$ is a complete isometry, the desired conclusion follows.

Let $n \in \mathbb{N}$ be fixed. We need to prove that

$$\left\| (\tilde{Q}^p)_n : M_n(S_{p'}^w[E] \widehat{\otimes}_{\text{proj}} S_p[F]) \rightarrow M_n(E \widehat{\otimes}_{d_p^c} F) \right\| \leq 1.$$

Denote by τ^k and $\tilde{\tau}^k$ the truncation mappings defined before the theorem and recall that, given $r \in \mathbb{N}$ and $Y \in M_r(S_p[F])$, by [33, Lemma 1.12], it holds

$$\left\| (\tilde{\tau}^k)_r(Y) - Y \right\|_{M_r(S_p[F])} \xrightarrow{k \rightarrow \infty} 0.$$

For $A \in M_n(S_{p'}^w[E] \widehat{\otimes}_{\text{proj}} S_p[F])$ take any representation of the form $A = \alpha(X \otimes Y)\beta$, with $X \in M_q(S_{p'}^w[E])$, $Y \in M_r(S_p[F])$, $\alpha \in M_{n,q,r}$, $\beta \in M_{q,r,n}$. By arguing as in the paragraph preceding the statement of the theorem, we obtain

$$(Q^p)_n(\tau^k \otimes \tilde{\tau}^k)_n(A) = (\tilde{Q}^p)_n(\tau^k \otimes \tilde{\tau}^k)_n(A) \xrightarrow{k \rightarrow \infty} (\tilde{Q}^p)_n(A), \text{ in } M_n(E \widehat{\otimes}_{d_p^c} F).$$

Also, the fact that Q^p is a complete 1-quotient implies, for each k ,

$$\begin{aligned} \left\| (Q^p)_n(\tau^k \otimes \tilde{\tau}^k)_n(A) \right\|_{M_n(E \widehat{\otimes}_{d_p^c} F)} &= \left\| (Q^p)_n(\alpha(\tau_q^k(X) \otimes \tilde{\tau}_r^k(Y))\beta) \right\|_{M_n(E \widehat{\otimes}_{d_p^c} F)} \\ &\leq \|\alpha\|_{M_{n,q,r}} \cdot \|\tau_q^k(X)\|_{M_q(S_{p'}^w \otimes_{\min} E)} \cdot \|\tilde{\tau}_r^k(Y)\|_{M_r(S_p[F])} \\ &\quad \cdot \|\beta\|_{M_{q,r,n}} \\ &\leq \|\alpha\|_{M_{n,q,r}} \cdot \|X\|_{M_q(S_{p'}^w[E])} \cdot \|\tilde{\tau}_r^k(Y)\|_{M_r(S_p[F])} \\ &\quad \cdot \|\beta\|_{M_{q,r,n}}. \end{aligned}$$

Taking limit as $k \rightarrow \infty$ we obtain

$$\left\| (\tilde{Q}^p)_n(A) \right\|_{M_n(E \widehat{\otimes}_{d_p^c} F)} \leq \|\alpha\|_{M_{n,q,r}} \cdot \|X\|_{M_q(S_{p'}^w[E])} \cdot \|Y\|_{M_r(S_p[F])} \cdot \|\beta\|_{M_{q,r,n}}.$$

Since this holds for every representation of A it is clear that

$$\left\| (\tilde{Q}^p)_n(A) \right\|_{M_n(E \widehat{\otimes}_{d_p^c} F)} \leq \|A\|_{M_n(S_{p'}^w[E] \widehat{\otimes}_{\text{proj}} S_p[F])}. \quad \square$$

Two direct consequences of the previous theorem are stated in the next corollary. Recall that limit in the first item below exists by the discussion immediately preceding Theorem 2.5.

Corollary 2.6. *Let $1 \leq p \leq \infty$ then:*

(1) The bilinear mapping associated to \widetilde{Q}^p , $B^p : S_{p'}^w[E] \times S_p[F] \rightarrow E \widehat{\otimes}_{d_p^o} F$ given by

$$B^p((x_{ij}), (y_{ij})) = \lim_{n \rightarrow \infty} \sum_{i,j=1}^n x_{ij} \otimes y_{ij}$$

is jointly completely bounded with $\|B^p\|_{\text{jcb}} = 1$.

(2) For each $u \in E \widehat{\otimes}_{d_p^o} F$,

$$\|u\|_{d_p^o} = \inf \left\{ \|(x_{ij})\|_{S_{p'}^w[E]} \|(y_{ij})\|_{S_p[F]} : u = \widetilde{Q}^p((x_{ij}) \otimes (y_{ij})) \right\}.$$

Now we show that certain two-sided multiplication operators are completely right p -nuclear. This is closely related to the results in [25], and the argument here follows closely [12, Prop. 12.2.2]. As in [33, p. 21], if E is an operator space, $x \in M_\infty(E)$ and $a \in M_\infty$, we denote by $a \cdot x$ (resp. $x \cdot a$) the matrix product, that is,

$$(a \cdot x)_{ij} = \sum_k a_{ik} x_{kj}, \quad \left(\text{resp. } (x \cdot a)_{ij} = \sum_k x_{ik} a_{kj} \right).$$

If $b \in M_\infty$, we denote $a \cdot x \cdot b = a \cdot (x \cdot b) = (a \cdot x) \cdot b$.

Proposition 2.7. *Let $1 \leq p \leq \infty$ and $a, b \in S_{2p}$. Then the multiplication operator $M(a, b) : x \mapsto a \cdot x \cdot b$ belongs to $\mathcal{N}_o^p(S_{p'}, S_1)$ and satisfies $\nu_o^p(M(a, b)) \leq \|a\|_{S_{2p}} \|b\|_{S_{2p}}$.*

Proof. Write $a = (a_{ij})$ and $b = (b_{ij})$. Let us first observe that we may assume that only finitely many entries in each of these two infinite matrices are nonzero. To that effect, suppose that we know the result for finitely supported matrices, and let $a, b \in S_{2p}$. We can find sequences $(a^k), (b^k)$ in S_{2p} of finitely supported matrices converging to a and b , respectively, in the S_{2p} norm. Since for $x \in S_{p'}$ we have

$$a^k x b^k - a^l x b^l = (a^k - a^l) x b^k + a^l x (b^k - b^l),$$

it follows that $(M(a^k, b^k))_k$ is a Cauchy sequence in $\mathcal{N}_o^p(S_{p'}, S_1)$ and therefore converges in the same space. Since the norm in $\mathcal{CB}(S_{p'}, S_1)$ is dominated by that of $\mathcal{N}_o^p(S_{p'}, S_1)$, it follows that the limit of $(M(a^k, b^k))_k$ has to be $M(a, b)$, and therefore $M(a, b) \in \mathcal{N}_o^p(S_{p'}, S_1)$ with $\nu_o^p(M(a, b)) \leq \|a\|_{S_{2p}} \|b\|_{S_{2p}}$.

Therefore, from now on we assume $a = (a_{ij})_{i,j=1}^n$ and $b = (b_{ij})_{i,j=1}^n$. Let $\varepsilon = [\varepsilon_{ij}]_{i,j=1}^n$ be the matrix of matrix units. For $x = (x_{ij}) \in S_{p'}$,

$$\begin{aligned} (axb)_{ij} &= \sum_{k,l} \varepsilon_{ij} \otimes a_{ik} x_{kl} b_{lj} \\ &= \sum_{k,l} \varepsilon_{ij} \otimes a_{ik} \varepsilon_{kl}(x) b_{lj}. \end{aligned}$$

From the above calculation

$$M(a, b)(x) = \tilde{Q}^p(\varepsilon \otimes (a \cdot \varepsilon \cdot b))(x),$$

hence $M(a, b) = \tilde{Q}^p(\varepsilon \otimes (a \cdot \varepsilon \cdot b))$ and

$$\nu_o^p(M(a, b)) \leq \|\varepsilon\|_{S_{p'}^w[S_p]} \|a \cdot \varepsilon \cdot b\|_{S_p[S_1]}.$$

Note that

$$\varepsilon = [\varepsilon_{ij}]_{i,j=1}^n \in S_{p'}^w[S_p] = \mathcal{CB}(S_p, S_p)$$

is just the projection onto an initial block S_p^n , and thus $\|\varepsilon\|_{S_{p'}^w[S_p]} = 1$.

On the other hand, by [33, Thm. 1.5 and Lemma 1.6],

$$\|a \cdot \varepsilon \cdot b\|_{S_p^n[S_1]} \leq \|a\|_{S_{2p}^n} \|\varepsilon\|_{M_n(S_1)} \|b\|_{S_{2p}^n}.$$

Now,

$$\varepsilon = [\varepsilon_{ij}]_{i,j=1}^n \in M_n(S_1) = M_n(S'_\infty) \subseteq \mathcal{CB}(S_\infty, M_n)$$

is once again just a projection onto an initial block, so $\|\varepsilon\|_{M_n(S_1)} = 1$. Therefore, we conclude that $\nu_o^p(M(a, b)) \leq \|a\|_{S_{2p}} \|b\|_{S_{2p}}$. \square

The multiplication operators defined in the previous proposition are the canonical prototypes of completely right p -nuclear mappings: we show below that a mapping is completely right p -nuclear if and only if it admits a factorization through one such multiplication operator (similar to the Banach space framework as in (3)).

Theorem 2.8. *For a linear map $T : E \rightarrow F$ and $1 \leq p \leq \infty$, the following are equivalent:*

- (a) T is completely right p -nuclear.
- (b) There exist $a, b \in S_{2p}$ such that T admits a factorization

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ U \downarrow & & \uparrow V \\ S_{p'} & \xrightarrow{M(a,b)} & S_1 \end{array}$$

Moreover, in this case

$$\nu_o^p(T) = \inf \{ \|U\|_{cb} \|V\|_{cb} \|a\|_{S_{2p}} \|b\|_{S_{2p}} \}$$

where the infimum is taken over all factorizations as in (b).

Proof. (b) \Rightarrow (a): This follows from Propositions 2.2 and 2.7.

(a) \Rightarrow (b): Let $1 \leq p < \infty$, assume that $T \in \mathcal{N}_o^p(E; F)$ with $\nu_o^p(T) < 1$. Then there exists $u \in E' \widehat{\otimes}_{d_p} F$ such that $J^p(u) = T$ and $\|u\|_{d_p} < 1$. By Theorem 2.3, we can in turn find $X \in S_{p'} \widehat{\otimes}_{\min} E'$, $Y \in S_p[F]$ such that $u = Q^p(X \otimes Y)$, $\|X\|_{S_{p'} \widehat{\otimes}_{\min} E'} < 1$ and $\|Y\|_{S_p[F]} < 1$. On the one hand, the representation of the minimal tensor product and its symmetry allows us to think of X as a mapping U in $\mathcal{CB}(E, S_{p'})$ with $\|U\|_{\text{cb}} < 1$. On the other hand, by [33, Thm. 1.5], we can write $Y = a \cdot \bar{Y} \cdot b$ with $\|a\|_{S_{2p}} < 1$, $\|b\|_{S_{2p}} < 1$ and $\|\bar{Y}\|_{S_\infty[F]} < 1$. Also, since $S_\infty[F] = S_\infty \widehat{\otimes}_{\min} F$ completely isometrically embeds into $\mathcal{CB}(S_1, F)$, \bar{Y} canonically induces a linear map $V : S_1 \rightarrow F$ with $\|V\|_{\text{cb}} = \|\bar{Y}\|_{S_\infty[F]}$. Chasing down the formulas it is easy to see that $T = V \circ M(a, b) \circ U$ (in the spirit of [12, Prop. 12.2.3]), giving us the desired factorization. Finally, for $p = \infty$ this is derived directly from the definition of right ∞ -nuclear with $M(a, b) = Id$. \square

3. Operator p -compact mappings

Following [37,9] we say that a linear mapping between Banach spaces $T : \mathbf{X} \rightarrow \mathbf{Y}$ is p -compact if $T(B_{\mathbf{X}})$ is a relatively p -compact set. That is, there exists a sequence $(y_n)_n \in \mathbf{Y}$ such that

$$T(B_{\mathbf{X}}) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : \sum_{n=1}^{\infty} |\alpha_n|^{p'} \leq 1 \right\} \text{ and } \sum_{n=1}^{\infty} \|y_n\|^p < \infty. \tag{6}$$

The class of p -compact mappings is denoted by \mathcal{K}_p , and endowed with the norm

$$\kappa_p(T) := \inf \{ \|(y_n)_n\|_{\ell_p(\mathbf{Y})} \}, \tag{7}$$

where the infimum runs all over the sequences $(y_n)_n \in \mathbf{Y}$ as in Equation (6).

This notion should not be confused with the homonymous concept studied in the late seventies and early eighties by [30] and [15] (see also [8]). Nowadays, this older class is sometimes referred to as “classical p -compact” mappings (as proposed in [27]). We will not deal with such mappings in this paper.

The following characterization of p -compactness in terms of commutative diagrams is well-known to experts, but we were unable to find an explicit reference for it. Thus, we include a sketch of its proof for completeness.

Proposition 3.1. *Let \mathbf{X} and \mathbf{Y} be Banach spaces, and $T : \mathbf{X} \rightarrow \mathbf{Y}$ a linear mapping. The following are equivalent:*

- (1) $T : \mathbf{X} \rightarrow \mathbf{Y}$ is p -compact

(2) There exist a Banach space \mathbf{Z} , a mapping $\Theta \in \mathcal{N}^p(\mathbf{Z}; \mathbf{Y})$ and a bounded mapping $R \in \mathcal{L}(\mathbf{X}, \mathbf{Z}/\ker \Theta)$ with $\|R\| \leq 1$ such that the following diagram commutes

$$\begin{array}{ccc}
 \mathbf{X} & \xrightarrow{T} & \mathbf{Y} & \xleftarrow{\Theta} & \mathbf{Z}, \\
 & \searrow R & \uparrow \tilde{\Theta} & \swarrow \pi & \\
 & & \mathbf{Z}/\ker \Theta, & &
 \end{array} \tag{8}$$

where π and $\tilde{\Theta}$ stand for the natural quotient mapping and the natural monomorphism associated to Θ respectively.

Moreover,

$$\kappa_p(T) = \inf\{\nu^p(\Theta) : \Theta \in \mathcal{N}^p(\mathbf{Z}; \mathbf{Y}) \text{ as in (8)}\}.$$

Proof. (1) \Rightarrow (2): This is essentially contained in the proof of [16, Prop. 2.9.] (which is based on [37, Thm. 3.2.]) but we anyway outline the idea. Given $\varepsilon > 0$, let $(y_n)_n \in \mathbf{Y}$ such that

$$T(B_{\mathbf{X}}) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : \sum_{n=1}^{\infty} |\alpha_n|^{p'} \leq 1 \right\} \text{ and } \left(\sum_{n=1}^{\infty} \|y_n\|^p \right)^{1/p} < \kappa_p(T)(1 + \varepsilon). \tag{9}$$

Then, we can find a factorization

$$\begin{array}{ccc}
 \mathbf{X} & \xrightarrow{T} & \mathbf{Y} & \xleftarrow{\Theta} & \ell_{p'} , \\
 & \searrow R & \uparrow \tilde{\Theta} & \swarrow \pi & \\
 & & \ell_{p'}/\ker \Theta, & &
 \end{array}$$

where $\Theta((\alpha_n)_n) = \sum_n \alpha_n y_n$ and for $x \in B_{\mathbf{X}}$ the element $R(x)$ in $\ell_{p'}/\ker \Theta$ is the class of the sequence $(\alpha_n(x))_n$ such that $Tx = \sum_n \alpha_n(x)y_n$. It is easy to see that Θ and R are well defined, $\|R\| \leq 1$ and Θ is p -nuclear. Also, $\nu^p(\Theta) \leq (\sum_{n=1}^{\infty} \|y_n\|^p)^{1/p} \leq \kappa_p(T)(1 + \varepsilon)$. So, item (2) holds with $\mathbf{Z} = \ell_{p'}$ and

$$\inf\{\nu^p(\Theta) : \Theta \in \mathcal{N}^p(\ell_{p'}; \mathbf{Y}) \text{ as in (8)}\} \leq \kappa_p(T).$$

(2) \Rightarrow (1): Given $\varepsilon > 0$ we have the following representation for Θ :

$$\Theta = \sum_{n \in \mathbb{N}} x'_n \otimes y_n,$$

where $(x'_n)_{n \in \mathbb{N}} \in \ell_{p'}^w(\mathbf{Z}')$, $(y_n)_{n \in \mathbb{N}} \in \ell_p(\mathbf{Y})$ and $\|(x'_n)\|_{\ell_{p'}^w(\mathbf{Z}')} \cdot \|(y_n)\|_{\ell_p(\mathbf{Y})} \leq (1 + \varepsilon)\nu^p(\Theta)$.

Let us show that $T = \tilde{\Theta}R$ is p -compact. Indeed, if $x \in B_X$ then $\|Rx\|_{\mathbf{Z}/\ker \Theta} \leq 1$. Thus, there is $z \in \mathbf{Z}$ such that $\pi z = Rx$, $\|z\|_{\mathbf{Z}} \leq 1 + \varepsilon$. Then, $Tx = \tilde{\Theta}Rx = \tilde{\Theta}\pi z = \Theta z = \sum_{n \in \mathbb{N}} x'_n(z)y_n$. Since

$$\|(x'_n(z))\|_{\ell_{p'}} \leq \|(x'_n)\|_{\ell_{p'}^w(\mathbf{Z}')} \|z\|_{\mathbf{Z}} \leq (1 + \varepsilon)\|(x'_n)\|_{\ell_{p'}^w(\mathbf{Z}')},$$

this shows that Tx lies in the p -convex hull of the sequence $(y_n \cdot (1 + \varepsilon)\|(x'_n)\|_{\ell_{p'}^w(\mathbf{Z}')})_{n \in \mathbb{N}}$. This implies that T is p -compact and moreover, $\kappa(T) \leq (1 + \varepsilon)^2 \nu^p(\Theta)$. \square

We now look for a definition of p -compact mappings in the noncommutative framework. We choose to define it as in Equation (8) since this will allow us to provide the ideal with an operator space structure. Later on, we will see an equivalent description which is more obviously similar to the original Banach space definition (see Theorem 3.11).

Definition 3.2. Let E and F be operator spaces. A mapping $T \in \mathcal{CB}(E, F)$ is called *operator p -compact* if there exist an operator space G , a completely right p -nuclear mapping $\Theta \in \mathcal{N}_o^p(G, F)$ and a completely bounded mapping $R \in \mathcal{CB}(E, G/\ker \Theta)$ with $\|R\|_{\text{cb}} \leq 1$ such that the following diagram commutes

$$\begin{array}{ccccc}
 E & \xrightarrow{T} & F & \xleftarrow{\Theta} & G \\
 & \searrow R & \uparrow \tilde{\Theta} & \swarrow \pi & \\
 & & G/\ker \Theta & &
 \end{array} \tag{10}$$

where π and $\tilde{\Theta}$ stand for the natural 1-quotient mapping and the natural monomorphism associated to Θ respectively.

The set of all operator p -compact mappings from E to F is denoted by $\mathcal{K}_p^o(E, F)$. For $T \in \mathcal{K}_p^o(E, F)$, we also define

$$\kappa_p^o(T) := \inf\{\nu_o^p(\Theta)\},$$

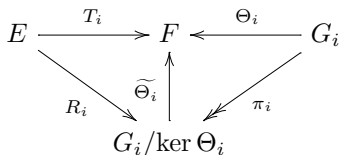
where the infimum runs over all possible completely right p -nuclear mappings $\Theta \in \mathcal{N}_o^p(G, F)$ as in (10).

Proposition 3.3. *Let E and F be operator spaces. The set $\mathcal{K}_p^o(E, F)$ is a linear subspace of $\mathcal{CB}(E, F)$ and κ_p^o defines a norm for this space.*

Proof. It is clear that $\kappa_p^o(T) \geq 0$, for all $T \in \mathcal{K}_p^o(E, F)$. Also, $\kappa_p^o(T) = 0$ implies, for each $\varepsilon > 0$, the existence of a commutative diagram as (10) with $\nu_o^p(\Theta) < \varepsilon$. Then, $\|\Theta\|_{\text{cb}} < \varepsilon$ and so $\|\tilde{\Theta}\|_{\text{cb}} < \varepsilon$ from which it is derived that $T = 0$.

If $T \in \mathcal{K}_p^o(E, F)$ and λ is a scalar, it is easy to see that $\lambda T \in \mathcal{K}_p^o(E, F)$ with $\kappa_p^o(\lambda T) = |\lambda| \kappa_p^o(T)$.

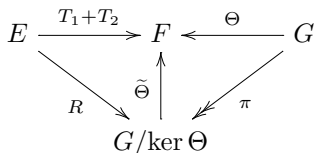
Now, let us consider $T_1, T_2 \in \mathcal{K}_p^o(E, F)$. For each $i = 1, 2$ we have a commutative diagram:



where G_i is an operator space, $R_i \in \mathcal{CB}(E, G_i)$ with $\|R_i\|_{cb} \leq 1$ and $\Theta_i \in \mathcal{N}_o^p(G_i, F)$. Define $G := G_1 \oplus_\infty G_2$, where $G_1 \oplus_\infty G_2$ stands for the direct sum of G_1 and G_2 (see the first two paragraphs of [34, Sec. 2.6]) and consider $\Theta : G \rightarrow F$ given by $\Theta(g_1, g_2) = \Theta_1 g_1 + \Theta_2 g_2$. If we denote, for each $i = 1, 2$, $\rho_i : G \rightarrow G_i$ the canonical restriction mapping we can write $\Theta = \Theta_1 \rho_1 + \Theta_2 \rho_2$. Since \mathcal{N}_o^p is a mapping ideal, we derive that $\Theta \in \mathcal{N}_o^p(G, F)$ with $\nu_o^p(\Theta) \leq \nu_o^p(\Theta_1) + \nu_o^p(\Theta_2)$.

We define $\Lambda : G_1 / \ker \Theta_1 \oplus_\infty G_2 / \ker \Theta_2 \rightarrow G / \ker \Theta$ by $\Lambda(\pi_1 g_1, \pi_2 g_2) = \pi(g_1, g_2)$, where $\pi : G \rightarrow G / \ker \Theta$ is the natural 1-quotient mapping. It is clear that Λ is well defined and (due to the operator space structure of the direct sum \oplus_∞) that it is completely bounded with $\|\Lambda\|_{cb} \leq 1$. The same is true for the mapping $R : E \rightarrow G / \ker \Theta$, given by $Rx = \Lambda(R_1 x, R_2 x)$, for any $x \in E$.

Now, we can assemble the following diagram:



Note that for each $x \in E$ and $i = 1, 2$ we have $R_i x = \pi_i g_i$ for certain $g_i \in G_i$. Then,

$$\begin{aligned}
 \tilde{\Theta}(Rx) &= \tilde{\Theta}(\Lambda(R_1 x, R_2 x)) = \tilde{\Theta}(\Lambda(\pi_1 g_1, \pi_2 g_2)) = \tilde{\Theta}(\pi(g_1, g_2)) \\
 &= \Theta(g_1, g_2) = \Theta_1 g_1 + \Theta_2 g_2 = \tilde{\Theta}_1(\pi_1 g_1) + \tilde{\Theta}_2(\pi_2 g_2) \\
 &= \tilde{\Theta}_1(R_1 x) + \tilde{\Theta}_2(R_2 x) = (T_1 + T_2)x.
 \end{aligned}$$

Therefore, the diagram is commutative and hence $T_1 + T_2$ belongs to $\mathcal{K}_p^o(E, F)$. Also, $\kappa_p^o(T_1 + T_2) \leq \nu_o^p(\Theta) \leq \nu_o^p(\Theta_1) + \nu_o^p(\Theta_2)$, for every Θ_1 and Θ_2 “admissible” for the factorization of T_1 and T_2 . Therefore, $\kappa_p^o(T_1 + T_2) \leq \kappa_p^o(T_1) + \kappa_p^o(T_2)$. \square

From the definition, it is straightforward to show that any $T \in \mathcal{N}_o^p(E, F)$ belongs to $\mathcal{K}_p^o(E, F)$ with $\kappa_p^o(T) \leq \nu_o^p(T)$.

We are now interested in giving an operator space structure to \mathcal{K}_p^o based on the relation, in the Banach space setting, of the right p -nuclear and p -compact ideals.

Definition 3.4. A mapping ideal \mathcal{A} is *surjective* if for each completely 1-quotient mapping $Q : G \twoheadrightarrow E$ we have $T \in \mathcal{A}(E, F)$ whenever $TQ \in \mathcal{A}(G, F)$ and the following condition is fulfilled $\|T\|_{\mathcal{A}} = \|TQ\|_{\mathcal{A}}$ for every matrix $T \in M_n(\mathcal{A}(E, F))$.

Definition 3.5. Given a mapping ideal \mathcal{A} , its *surjective hull* \mathcal{A}^{sur} is the smallest surjective mapping ideal that contains \mathcal{A} .

An operator space E is called projective if, for any $\varepsilon > 0$, any completely bounded map $T : E \rightarrow F/S$ into a quotient space (here F is any operator space and $S \subset F$ any closed space) admits a lifting $\tilde{T} : E \rightarrow F$ with $\|\tilde{T}\|_{cb} \leq (1 + \varepsilon)\|T\|_{cb}$.

We recall [34, Prop. 2.12.2] that for every operator space E there is a set I and a family $(n_i)_{i \in I} \subset \mathbb{N}$ such that E is the quotient of $\ell_1(\{S_1^{n_i} : i \in I\})$. We denote the latter space by Z_E and $Q_E : Z_E \twoheadrightarrow E$ the corresponding completely 1-quotient mapping. The space Z_E is projective (see for example [34, Chap. 24]).

Proposition 3.6. *For every E and F operator spaces we have the following characterization of the surjective hull*

$$\mathcal{A}^{sur}(E, F) = \{T \in \mathcal{CB}(E, F) : TQ_E \in \mathcal{A}(E, F)\},$$

with $\|T\|_{\mathcal{A}^{sur}} = \|TQ_E\|_{\mathcal{A}}$.

Proof. We define

$$\mathcal{U}(E, F) = \{T \in \mathcal{CB}(E, F) : TQ_E \in \mathcal{A}(E, F)\},$$

endowed with the operator space norm given by $\|T\|_{\mathcal{U}} = \|TQ_E\|_{\mathcal{A}}$, for every $T \in M_n(\mathcal{U}(E, F))$.

We now show that \mathcal{U} is a mapping ideal. Indeed, if $T \in M_n(\mathcal{U}(E, F))$ we obviously have

$$\|T\|_{\mathcal{U}(E, F)} = \|TQ_E\|_{\mathcal{A}} \geq \|TQ_E\|_{cb} = \|T\|_{cb}.$$

Let $T \in M_n(\mathcal{U}(E, F))$, $R \in \mathcal{CB}(E_0, F)$, $S \in \mathcal{CB}(E, F_0)$. Since Z_{E_0} is projective, given $\varepsilon > 0$, there is a lifting $L_\varepsilon \in \mathcal{CB}(Z_{E_0}, Z_E)$ of RQ_{E_0} with $\|L_\varepsilon\|_{cb} \leq (1 + \varepsilon)\|R\|_{cb}$ such that the following diagram commutes

$$\begin{array}{ccccc} E_0 & \xrightarrow{R} & E & \xrightarrow{T_{i,j}} & F & \xrightarrow{S} & F_0 . \\ \uparrow Q_{E_0} & & \uparrow Q_E & & & & \\ Z_0 & \xrightarrow{L_\varepsilon} & Z_E & & & & \end{array}$$

Then, for every $1 \leq i, j \leq n$ we have $ST_{i,j}RQ_{E_0} = ST_{i,j}Q_E L_\varepsilon \in \mathcal{A}(Z_{E_0}, F_0)$ and hence $ST_{i,j}R \in \mathcal{U}(E_0, F_0)$. Moreover, by the ideal property of \mathcal{A} ,

$$\begin{aligned} \|S_n TR\|_{M_n(\mathcal{U}(E_0, F_0))} &= \|S_n TRQ_{E_0}\|_{M_n(\mathcal{A}(Z_{E_0}, F_0))} = \|S_n TQ_E L_\varepsilon\|_{M_n(\mathcal{A}(Z_{E_0}, F_0))} \\ &\leq \|S_n TQ_E\|_{M_n(\mathcal{A}(Z_E, F_0))} (1 + \varepsilon) \|R\|_{cb} \\ &\leq (1 + \varepsilon) \|S\|_{cb} \|T\|_{M_n(\mathcal{U}(E, F))} \|R\|_{cb}. \end{aligned}$$

We now prove that the mapping ideal \mathcal{U} is surjective. Let $Q : G \rightarrow E$ be a complete 1-quotient mapping and $T \in \mathcal{CB}(E, F)$ such that $TQ \in \mathcal{U}(G, F)$. Since Z_E is projective and $QQ_G : Z_G \rightarrow E$ is a complete 1-quotient mapping, given $\varepsilon > 0$, there is a lifting of $Q_E, L_\varepsilon : Z_E \rightarrow Z_G$ with cb-norm less than or equal to $1 + \varepsilon$.

We have $TQ_E = TQQ_G L_\varepsilon \in \mathcal{A}(Z_E, F)$ since $TQQ_G \in \mathcal{A}(Z_G, F)$. Then, $T \in \mathcal{U}(E, F)$. Also, if $T \in M_n(\mathcal{U}(E, F))$ we obtain

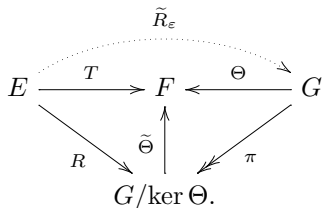
$$\begin{aligned} \|T\|_{\mathcal{U}} &= \|TQ_E\|_{\mathcal{A}} = \|TQQ_G L_\varepsilon\|_{\mathcal{A}} \leq (1 + \varepsilon) \|TQQ_G\|_{\mathcal{A}} \\ &= (1 + \varepsilon) \|TQ\|_{\mathcal{U}} \leq (1 + \varepsilon) \|T\|_{\mathcal{U}}. \end{aligned}$$

Note that, by the definition, $\mathcal{U}(E, F) \subset \mathcal{A}^{sur}(E, F)$: if $T \in \mathcal{U}(E, F)$, we have $TQ_E \in \mathcal{A}(Z_E, F) \subset \mathcal{A}^{sur}(Z_E, F)$. Since \mathcal{A}^{sur} is surjective we obtain that $T \in \mathcal{A}^{sur}(E, F)$.

We have shown that \mathcal{U} is a surjective mapping ideal (which obviously contains \mathcal{A}). Then, by minimality we have the reverse inclusion. \square

Proposition 3.7. *Let E be a projective operator space. Then, $T \in \mathcal{K}_p^o(E, F)$ if and only if $T \in \mathcal{N}_p^p(E, F)$ and $\kappa_p^o(T) = \nu_p^p(T)$.*

Proof. Let $T \in \mathcal{K}_p^o(E, F)$, and consider a factorization as in (10). Given $\varepsilon > 0$, since E is projective, there is a lifting of $R, \tilde{R}_\varepsilon \in \mathcal{CB}(E, G)$ with $\|\tilde{R}_\varepsilon\| \leq 1 + \varepsilon$, such that the following diagram commutes



Then, $T = \tilde{\Theta} \tilde{R}_\varepsilon \in \mathcal{N}_p^p(E, F)$ and $\nu_p^p(T) \leq (1 + \varepsilon) \nu_p^p(\tilde{\Theta})$. Observe that this holds for every $\tilde{\Theta}$ verifying Equation (10). Thus, by definition of the norm $\kappa_p^o(T)$, we obtain $\nu_p^p(T) \leq (1 + \varepsilon) \kappa_p^o(T)$ and the result follows. \square

Proposition 3.8. *Let E and F be operator spaces. Then $T \in \mathcal{K}_p^o(E, F)$ if and only if $TQ_E \in \mathcal{N}_p^p(Z_E, F)$ and $\kappa_p^o(T) = \nu_p^p(TQ_E)$.*

Proof. If $T \in \mathcal{K}_p^o(E, F)$, then $TQ_E \in \mathcal{K}_p^o(Z_E, F)$. Now by Proposition 3.7, $TQ_E \in \mathcal{N}_o^p(Z_E, F)$ and also $\nu_o^p(TQ_E) = \kappa_p^o(TQ_E) \leq \kappa_p^o(T)$, since Z_E is projective.

Reciprocally, if $TQ_E \in \mathcal{N}_o^p(Z_E, F)$, we define $R \in \mathcal{CB}(E, Z_E/\ker \Theta)$ in the following way: $Rx := \pi y$ where $Q_E y = x$. It is not difficult to check that R is well defined and also $\|R\|_{cb} \leq 1$. If we denote $\Theta := TQ_E$ we have $\tilde{\Theta}Rx = \tilde{\pi}y = \Theta y = TQ_E y = Tx$ for every $x \in E$. Thus, the following diagram commutes

$$\begin{array}{ccccc}
 E & \xrightarrow{T} & F & \xleftarrow{\Theta} & Z_E \\
 & \searrow R & \uparrow \tilde{\Theta} & \swarrow \pi & \\
 & & Z_E/\ker \Theta & &
 \end{array}$$

Therefore, T is an operator p -compact mapping and $\kappa_p^o(T) \leq \nu_o^p(TQ_E)$. This concludes the proof. \square

We have shown that for every E and F operator spaces, we have the equality (as Banach spaces)

$$\mathcal{K}_p^o(E, F) = (\mathcal{N}_o^p)^{sur}(E, F).$$

This induces a natural operator space structure for $\mathcal{K}_p^o(E, F)$. Indeed, if $T \in M_n(\mathcal{K}_p^o(E, F))$, we define

$$\|T\|_{M_n(\mathcal{K}_p^o(E, F))} := \|T\|_{M_n((\mathcal{N}_o^p)^{sur}(E, F))} = \|TQ_E\|_{M_n(\mathcal{N}_o^p(Z_E, F))}.$$

As a consequence we can say that \mathcal{K}_p^o is a mapping ideal through the following identification:

$$\mathcal{K}_p^o = (\mathcal{N}_o^p)^{sur}. \tag{11}$$

In the Banach space setting, a continuous linear mapping is compact if it sends the unit ball into a relatively compact set. Equivalently, it sends the unit ball into the closure of the convex hull of a null sequence. In the operator space framework, if we look at how a mapping acts on the “matrix unit ball” (see definition below) the previous conditions are not equivalent. Webster [38,39] noticed this and studied several notions of compactness: operator compactness, strong operator compactness, and matrix compactness. Let us recall the definition of “operator compactness” that later we will naturally extend to the case of p -compactness.

If E is an operator space, a *matrix set* is a sequence of sets $\mathbf{K} = (K_n)$, where $K_n \subset M_n(E)$, for all n . The closure of a matrix set means the closure of each set of the sequence in the corresponding matrix space. The *matrix unit ball* is the matrix set $(B_{M_n(E)})$.

Given $X \in S_\infty[E]$, Webster defined the *absolutely matrix convex hull* of X to be the matrix set $\mathbf{co}(X)$ where

$$(\mathbf{co}(X))_k = \{v \in M_k(E) : \exists \sigma \in M_k(M_\infty^{fin}), \|\sigma\|_{M_k(S_1)} \leq 1 \text{ such that } (\sigma \otimes id)X = v\}.$$

A mapping $T \in \mathcal{CB}(E, F)$ is *operator compact* if the image by T of the matrix unit ball of E is contained in $\overline{\mathbf{co}(Y)}$, for some $Y \in \mathcal{S}_\infty[F]$.

Based on the work of Webster we now define the notion of p -absolutely matrix convex hull.

Definition 3.9. Let E be an operator space and $X \in S_p[E]$, for $1 \leq p \leq \infty$, we define the p -absolutely matrix convex hull of X to be the matrix set $\mathbf{co}_p(X)$ where

$$(\mathbf{co}_p(X))_k = \{v \in M_k(E) : \exists \sigma \in M_k(M_\infty^{fin}), \|\sigma\|_{M_k(S_{p'})} \leq 1 \text{ such that } (\sigma \otimes id)X = v\}.$$

We say that a matrix set $\mathbf{K} = (K_n)$ in E is *operator p -compact* if \mathbf{K} is contained in $\overline{\mathbf{co}_p(X)}$, for some $X \in S_p[E]$.

We are using the term “operator p -compact” for two seemingly different notions, but they will turn out to be the same. If we denote by \mathbf{B}_E the matrix unit ball of E , we will prove that $T : E \rightarrow F$ is *operator p -compact* if and only if the image under T of the matrix unit ball of E is an operator p -compact matrix set in F .

Note that the ∞ -absolutely matrix convex hull is Webster’s absolutely matrix convex hull, and thus operator ∞ -compact mappings are the same as operator compact mappings.

Remark 3.10. Given $1 \leq p \leq \infty$, for any $X \in S_p[E]$ and $k \in \mathbb{N}$ we have

$$\left(\overline{\mathbf{co}_p(X)}\right)_k = \{v \in M_k(E) : \exists \sigma \in M_k(S_{p'}), \|\sigma\|_{M_k(S_{p'})} \leq 1 \text{ such that } (\sigma \otimes id)X = v\},$$

where, in the case $p = 1$ the space $S_{p'}$ should be replaced by M_∞ .

Proof. Indeed, let us denote by C_k the set on the right hand side. For $1 < p \leq \infty$, it is enough to show that C_k is closed. Consider $\Psi : M_k(S_{p'}) \rightarrow M_k(E)$ defined by $\sigma \mapsto (\sigma \otimes id)(X)$. Note that the image by Ψ of $B_{M_k(S_{p'})}$, the closed unit ball of $M_k(S_{p'})$, is exactly C_k . Take a sequence $(\sigma_l)_l \in B_{M_k(S_{p'})}$ such that $\Psi(\sigma_l) \rightarrow v \in M_k(E)$. We have to see that $v = \Psi(\sigma)$ for certain $\sigma \in M_k(S_{p'})$ with $\|\sigma\|_{M_k(S_{p'})} \leq 1$. Since $B_{M_k(S_{p'})}$ is weak* sequentially compact (since the predual of $M_k(S_{p'})$ is separable) there is a subsequence $(\sigma_{l_j})_j$ weak* convergent to an element $\sigma \in B_{M_k(S_{p'})}$. By uniqueness of the limit our conclusion follows once proving that for all matrices $v' \in M_k(E')$ we have the convergence of the scalar pairing (in the sense of [12, 1.1.24])

$$\langle \Psi(\sigma_{l_j}), v' \rangle \rightarrow \langle \Psi(\sigma), v' \rangle.$$

Now,

$$\langle \Psi(\sigma_{l_j}), v' \rangle = \langle (\sigma_{l_j} \otimes id)(X), v' \rangle = \langle \sigma_{l_j}, (id \otimes v')(X) \rangle \rightarrow \langle \sigma, (id \otimes v')(X) \rangle = \langle \Psi(\sigma), v' \rangle.$$

For $p = 1$, the previous argument (replacing $S_{p'}$ by M_∞) shows that $\left(\overline{\mathbf{co}_1(X)}\right)_k \subset C_k$. The other inclusion runs as follows. Recall, using the notation as in Theorem 2.5, that any $X \in S_1[E]$ can be approximated by its “truncations” $\{\tilde{\tau}^m(X)\}_m$ [33, Lemma 1.12] and denote by $\rho^m : M_\infty \rightarrow M_m$ the truncated mapping between these matrix spaces. Now, for $v = (\sigma \otimes id)X \in C_k$ with $\sigma \in M_k(M_\infty)$, we have to see that $\{((\rho^m)_k \sigma \otimes id)X\}_m \subset \mathbf{co}_1(X)$ approximates v . Indeed,

$$\begin{aligned} \|v - ((\rho^m)_k \sigma \otimes id)X\|_{M_k(E)} &= \|((\sigma - (\rho^m)_k \sigma) \otimes id)X\|_{M_k(E)} \\ &= \|(\sigma \otimes id)(X - (\tilde{\tau}^m)_k(X))\|_{M_k(E)} \\ &\leq \|X - (\tilde{\tau}^m)_k(X)\|_{M_k(S_1[E])} \xrightarrow{m \rightarrow \infty} 0. \quad \square \end{aligned}$$

The following result shows that the first definition of p -compactness (the one which involves the factorization through completely right p -nuclear mappings) and the definition based on Webster’s work are the same. We highlight the analogy presented between Equations (7) and (12), below.

Theorem 3.11. *Let $T : E \rightarrow F$ be a completely bounded mapping and $1 \leq p \leq \infty$. Then T is operator p -compact if and only if $T(\mathbf{B}_E)$ is an operator p -compact matrix set in F . Moreover,*

$$\kappa_p^o(T) = \inf\{\|Y\|_{S_p[F]} : T(\mathbf{B}_E) \subset \overline{\mathbf{co}_p(Y)}\}. \tag{12}$$

Proof. Suppose T is operator p -compact. Without loss of generality we may assume that $\kappa_p^o(T) < 1$. By the commutative diagram given in (10) there is a completely right p -nuclear mapping $\Theta : G \rightarrow F$ with ν_o^p -norm less than 1. It suffices to see that Θ is operator p -compact, since $T(\mathbf{B}_E)$ is contained in $\Theta(\mathbf{B}_G)$. By Theorem 2.3 we can write $\Theta = J^p \circ Q^p(X \otimes Y)$ where $\|X\|_{S_{p'} \widehat{\otimes}_{\min} G'} < 1$ and $\|Y\|_{S_p[F]} < 1$. For $g \in M_n(G)$ of norm less than one, we will show that $\Theta_n g \in \overline{(\mathbf{co}_p(Y))}_n$. Indeed, for $1 < p \leq \infty$, denoting by $\iota_G : G \rightarrow G''$ the canonical complete isometry, since $X \in S_{p'} \widehat{\otimes}_{\min} G'$ we derive $X' \circ \iota_G \in \mathcal{CB}(G, S_{p'})$. For $p = 1$, being $X \in S_\infty \widehat{\otimes}_{\min} G'$ we obtain $X' \circ \iota_G \in \mathcal{CB}(G, M_\infty)$. Consider $\sigma := (X' \circ \iota_G)_n g \in M_n(S_{p'})$ (replacing $S_{p'}$ by M_∞ in the case $p = 1$), then, $\|\sigma\|_{M_n(S_{p'})} < 1$ and it is not hard to check that

$$\Theta_n g = (\sigma \otimes id)(Y).$$

Then, $T(\mathbf{B}_E) \subset \Theta(\mathbf{B}_G) \subset \overline{\mathbf{co}_p(Y)}$ and hence $\inf\{\|Y\|_{S_p[F]} : T(\mathbf{B}_E) \subset \overline{\mathbf{co}_p(Y)}\} < 1$.

For the converse, let $T : E \rightarrow F$ be an operator p -compact mapping with $\inf\{\|Y\|_{S_p[F]} : T(\mathbf{B}_E) \subset \overline{\mathbf{co}_p(Y)}\} < 1$. We begin by considering the case $1 < p < \infty$. Denote by $\Theta : S_{p'} \rightarrow F$, the operator given by $J^p \circ B^p(id, Y)$, which by definition is in \mathcal{N}_o^p since $id \in S_{p'}^w[S_p] = \mathcal{CB}(S_p, S_{p'})$. Observe that by Corollary 2.6 (1), $\nu_o^p(\Theta) < 1$. Remark 3.10 asserts that for each $x \in E$ there is $\sigma \in S_{p'}$ such that $Tx = (\sigma \otimes id)(Y)$. Defining $Rx := [\sigma]$ we have the following commutative diagram

$$\begin{array}{ccccc}
 E & \xrightarrow{T} & F & \xleftarrow{\Theta} & S_{p'} \\
 & \searrow R & \uparrow \tilde{\Theta} & \swarrow \pi & \\
 & & S_{p'}/\ker \Theta & &
 \end{array}$$

Note that R is well defined. Moreover, if $x \in B_{M_k(E)}$ by hypothesis $T_k x = (\tilde{\sigma} \otimes id)(Y)$, for certain $\tilde{\sigma}$ with $\|\tilde{\sigma}\|_{M_k(S_{p'})} \leq 1$. Checking coordinates we have $T(x_{i_j}) = (\tilde{\sigma}_{i,j} \otimes id)(Y)$ and therefore $R(x_{i_j}) = [\tilde{\sigma}_{i,j}]$. Hence, $R_k x = ([\tilde{\sigma}_{i,j}]_{i,j})$ and then $\|R\|_{cb} \leq 1$. We conclude that $T \in \mathcal{K}_p^o(E; F)$ and $\kappa_p^o(T) \leq \nu_p^o(\Theta) < 1$.

For $p = 1$, we have $Y \in S_1[F] = S_1 \widehat{\otimes}_{proj} F$. Since the projective tensor norm satisfies the Embedding Lemma (see [11] or [6]), there is a completely isometric embedding $\kappa : S_1 \widehat{\otimes}_{proj} F \rightarrow S_1'' \widehat{\otimes}_{proj} F$. Now, $\Theta = J^1 \circ \kappa(Y)$ belongs to $\mathcal{N}_o^1(M_\infty, F)$, where $J^1 : S_1'' \widehat{\otimes}_{proj} F \rightarrow S_1'' \widehat{\otimes}_{min} F$ is the canonical mapping. The rest of the argument follows as in the previous case.

For $p = \infty$, we have $Y \in S_\infty[F]$ and so $B^\infty(id, Y)$ belongs to $S_\infty \widehat{\otimes}_{d_\infty} F$. Since d_∞^o is finitely generated and S_∞ is locally reflexive, we can appeal again to the Embedding Lemma [6] to get a completely isometric embedding $\kappa : S_\infty \widehat{\otimes}_{d_\infty} F \rightarrow M_\infty \widehat{\otimes}_{d_\infty} F$. Hence, $\Theta = J^\infty \circ \kappa \circ B^\infty(id, Y)$ belongs to $\mathcal{N}_o^\infty(S_1, F)$ and the proof finishes as in the two previous cases. \square

The previous theorem, together with the relation presented in Equation (11), allows us to endow Webster’s class (of operator compact mappings) with a natural operator space structure, showing that this class is indeed a mapping ideal. Namely, it is exactly the ideal \mathcal{K}_∞^o .

4. Some final questions

We present some open questions regarding the mapping ideal \mathcal{K}_p^o .

In parallel to the Banach space case, we say that a mapping ideal $(\mathfrak{A}, \mathbf{A})$ and an operator space tensor norm α are *associated*, denoted $(\mathfrak{A}, \mathbf{A}) \sim \alpha$ if for every pair of finite-dimensional operator spaces (E, F) we have a complete isometry $\mathfrak{A}(E, F) = E' \otimes_\alpha F$, given by the canonical map. Notice that since this definition is based only on finite-dimensional spaces, two different mapping ideals can be associated to the same tensor norm (and vice versa). For example, from Definition 2.1 it is clear that d_p^o is associated to the mapping ideal \mathcal{N}_p^o .

Recall that an operator space tensor norm α is called left-injective if for any operator spaces E_1, E_2, F and a complete isometry $i : E_1 \rightarrow E_2$, the mapping

$$i \otimes id_F : E_1 \otimes_\alpha F \rightarrow E_2 \otimes_\alpha F$$

is completely isometric. Given an operator space tensor norm we denote by $/\alpha$ the greatest left-injective tensor norm that is dominated by α . This essentially mimics the

notion given in the context of normed spaces deeply described in [8, Chap. 20] and [36, Sec. 7.2].

In the Banach space setting, it is shown in [16, Thm. 3.3] that $/d_p$ is the tensor norm associated to \mathcal{K}_p . Therefore, is natural to ask if an analogous result holds in the operator space framework.

Question 4.1. *Is the operator space tensor norm $/d_p^o$ associated to the mapping ideal \mathcal{K}_p^o ?*

This question is a particular case of the following more general one (whose statement is valid in the Banach space context):

Question 4.2. *Let \mathcal{A} be a mapping ideal and α its associated operator space tensor norm. Is $/\alpha$ associated to \mathcal{A}^{sur} ?*

Note that $d_p^o \sim \mathcal{N}_o^p$ and, by Equation (11), we have $\mathcal{K}_p^o = (\mathcal{N}_o^p)^{sur}$. Thus an affirmative answer to this question would provide also an answer to the first one. Results of this kind for the Banach space setting, although sometimes hard to see it at first glance, are based on *local techniques* (e.g., [8, Prop. 20.9]) which are no longer valid in the operator space framework.

In [6] it is shown that if $\beta \sim \mathcal{A}^{sur}$ then β is left-injective. Therefore, since $\mathcal{A} \subset \mathcal{A}^{sur}$ we have that $\beta \leq / \alpha$. An affirmative answer to Question 4.2 is obtained in [6] for the particular case where α is left accessible (definition similar to the Banach space case). It is known in the Banach space setting that d_p satisfies this property ([8, Thm. 21.5], [36, Prop. 7.21.]), but unfortunately we do not know yet whether this also holds for the operator space counterpart d_p^o .

Another interesting question, which involves the mapping ideal \mathcal{K}_∞^o , was posed by Webster in his thesis [38, Sec. 4.1]:

Problem 4.3. Webster: “It is an open question as to whether the space of operator compact maps must be closed in the cb-norm topology”.

In other words, if a sequence of operator compact mappings $(T_n)_{n \in \mathbb{N}}$ converges to T in the completely bounded norm ($\|T - T_n\|_{cb} \rightarrow 0$), does it imply that T is also an operator compact map?

By Theorem 3.11, as a direct consequence of the Open Mapping Theorem we see that Problem 4.3 can be reformulated in terms of the following question.

Question 4.4. *Are the norms κ_∞^o and $\|\cdot\|_{cb}$ equivalent on \mathcal{K}_∞^o ?*

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