



p -Converging operators and Dunford–Pettis property of order p

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ABSTRACT

In the present paper, a new concept of weakly p -Cauchy sequences is introduced to study p -converging operators and Dunford–Pettis property of order p . The notion of p -(V) sets is defined to characterize Dunford–Pettis property of order p . We also introduce a complete locally convex topology by relatively weakly p -compact sets to quantify Dunford–Pettis property of order p .

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1. Introduction and notations

Throughout the paper, p^* denotes the conjugate number of p for $1 \leq p < \infty$; if $p = 1$, l_{p^*} plays the role of c_0 . By an operator, we always mean a bounded linear operator. Let X, Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of all operators from X into Y and $\mathcal{K}(X, Y)$ by the space of all the compact operators from X into Y . For a Banach space X , the closed unit ball of X is denoted by B_X and the identity map on X is denoted by I_X . Let X be a Banach space and $1 \leq p < \infty$. A sequence $(x_n)_n$ in X is *weakly p -summable* if the scalar sequence $(\langle x^*, x_n \rangle)_n$ is in l_p for every $x^* \in X^*$. For $p = 1$, weakly 1-summable sequences correspond to weakly unconditionally Cauchy series; For $p = \infty$, weakly ∞ -summable sequences are just bounded sequences. We denote by $l_p^w(X)$ the space of all weakly p -summable sequences in X , endowed with the norm

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$$\|(x_n)_n\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}, \quad (x_n)_n \in l_p^w(X).$$

Further, $l_{\infty}^w(X) = l_{\infty}(X)$. Following [17], we write $c_0^w(X)$ for weakly null sequences in X . Obviously, if $1 \leq p_1 < p_2 < \infty$, then $l_{p_1}^w(X) \subset l_{p_2}^w(X) \subset c_0^w(X)$. It is clear that the unit vector basis in l_{p^*} is weakly p -summable.

A sequence $(x_n)_n \in l_p^w(X)$ is *unconditionally p -summable* if

$$\sup\left\{\left(\sum_{n=m}^{\infty} |\langle x^*, x_n \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

It is noted that unconditionally 1-summable sequences are just unconditionally summable sequences and unconditionally ∞ -summable sequences are precisely norm null sequences. We denote the space of all unconditionally p -summable sequences in X by $l_p^u(X)$, which is a closed subspace of $l_p^w(X)$ under the norm $\|\cdot\|_p^w$. For $p = \infty$, the space $l_{\infty}^u(X)$ is identical to $c_0(X)$, the space of all norm null sequences in X . If $1 \leq p_1 < p_2 \leq \infty$, then $l_{p_1}^u(X) \subset l_{p_2}^u(X) \subset c_0(X)$. For the sake of notation, for $p = \infty$, we prefer to consider the space $c_0^w(X)$ instead of $l_{\infty}^u(X) = l_{\infty}(X)$ in the present paper.

It is a well-known result of A. Grothendieck ([19], [17, Proposition 2.2]) that the canonical correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{L}(l_{p^*}, X)$ onto $l_p^w(X)$ when $1 < p < \infty$; For $p = 1$, the isometric isomorphism is from $\mathcal{L}(c_0, X)$ onto $l_1^w(X)$; For $p = \infty$, the isometric isomorphism is from $\mathcal{L}(l_1, X)$ onto $l_{\infty}(X)$. J.H. Fourie and J. Swart ([18, Theorem 1.4(b)]) proved that the same correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{K}(l_{p^*}, X)$ onto $l_p^u(X)$ when $1 < p < \infty$; For $p = 1$, the isometric isomorphism is from $\mathcal{K}(c_0, X)$ onto $l_1^u(X)$; Nevertheless, the correspondence $T \mapsto (Te_n)_n$ does not provide an isometric isomorphism of $\mathcal{K}(l_1, X)$ onto $c_0(X)$. Indeed, it is easy to see that an operator T from l_1 to a Banach space X is compact if and only if the sequence $(Te_n)_n$ is relatively norm compact (see [16, p. 114]).

Let us recall that an operator $T : X \rightarrow Y$ is *unconditionally converging* if T takes weakly unconditionally Cauchy series to unconditionally convergent series. It can be proved that an operator T is unconditionally converging if and only if T maps weakly unconditionally Cauchy series to norm null sequences. Recall that an operator $T : X \rightarrow Y$ is *completely continuous* if T takes weakly null sequences to norm null sequences. In [4], J.M.F. Castillo introduced the ideal \mathcal{C}_p of p -converging operators, which is intermediate between the ideal \mathcal{C}_1 (unconditionally converging operators) and the ideal \mathcal{C}_{∞} (completely continuous operators). He defined the ideal of p -converging operators as those operators which transformed weakly p -summable sequences into norm null sequences, basing upon the following result (see the introduction of [5]):

$$\mathcal{L}(l_p, X) = \mathcal{K}(l_p, X) \text{ if and only if every weakly } p^* \text{-summable sequence in } X \text{ is norm null.}$$

In the introduction part of [5], J. M. F. Castillo also mentioned that p -converging operators received its name from the fact that \mathcal{C}_1 is precisely the ideal of unconditionally converging operators. By the above mentioned results of A. Grothendieck, J.H. Fourie and J. Swart, we see that $\mathcal{L}(l_p, X) = \mathcal{K}(l_p, X)$ if and only if every weakly p^* -summable sequence in X is unconditionally p^* -summable. Thus, p -converging operators are precisely those operators which transformed weakly p -summable sequences into unconditionally p -summable sequences. Since then, an extensive research on p -converging operators and their applications to polynomial forms are done in a series of papers of J.M.F. Castillo et al. (see [4,5,7–9,11] etc.).

The present paper is organized as follows.

It is a well-known basic fact that an operator $T : X \rightarrow Y$ is completely continuous if and only if T maps weakly Cauchy sequences to norm convergent sequences. To extend this elementary fact to p -converging operators, we introduce a new concept of weakly p -Cauchy sequences which is a generalization of classical

concept of weakly Cauchy sequences in Section 2. We prove that an operator $T : X \rightarrow Y$ is p -converging if and only if T maps weakly p -Cauchy sequences to norm convergent sequences. In this section, a counterexample is constructed to show that an operator is p -converging not precisely when its second adjoint is.

Section 3 is concerned with Dunford–Pettis property of order p (DPP_p for short) introduced in [9], which is a generalization of the classical Dunford–Pettis property. Let us recall that a Banach space X has the *Dunford–Pettis property* (in short, DPP) if for every Banach space Y , every weakly compact operator $T : X \rightarrow Y$ is completely continuous (see [20]). An operator $T : X \rightarrow Y$ is said to be *weakly compact* if TB_X is relatively weakly compact in Y . It turns out that many classical spaces failing DPP enjoy DPP_p , such as Hardy space H^1 and Lorentz function spaces $\Lambda(W, 1)$. It is a classical result that a Banach space X has the DPP if and only if $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$ for every weakly null sequence $(x_n)_n$ in X^* and every weakly Cauchy sequence $(x_n)_n$ in X ; It is proved in [9] that a Banach space X has the DPP_p if and only if $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$ for every weakly p -summable sequence $(x_n)_n$ in X and every weakly null sequence $(x_n^*)_n$ in X^* ; In this section, we use weakly p -Cauchy sequences and prove that a Banach space X has the DPP_p if and only if $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$ for every weakly p -Cauchy sequence $(x_n)_n$ in X and every weakly null sequence $(x_n^*)_n$ in X^* . New characterizations of DPP_p in dual spaces are obtained. Following [15], we say that a Banach space X has the hereditary Dunford–Pettis property if every closed subspace of X has the DPP . It is well-known (see [15, Theorem 6]) that if a Banach space X has the hereditary Dunford–Pettis property then X has the weak Banach–Saks property. Recall that a Banach space X has the *weak Banach–Saks property* if every weakly null sequence $(x_n)_n$ in X admits a subsequence $(x_{n_k})_k$ such that the Cesàro means $\frac{1}{N} \sum_{k=1}^N x_{n_k}$ converges in norm. It is natural to generalize hereditary Dunford–Pettis property to more general case p . In this section, we also introduce the notion of hereditary Dunford–Pettis property of order p and establish its characterizations. These characterizations are p -versions of the standard characterizations of hereditary Dunford–Pettis property. Moreover, we introduce the notion of weak Banach–Saks property of order p and prove that if a Banach space X has the hereditary DPP_p then X has the weak Banach–Saks property of order p . This result is a p -version of the above famous result. Localizing the original definition of DPP , a bounded subset A of a Banach space X is called a *Dunford–Pettis set* if for all Banach spaces Y every weakly compact operator from X to Y maps A onto a relatively norm compact set. Consequently, a Banach space X has the DPP if and only if every relatively weakly compact subset of X is a Dunford–Pettis set. In [1], K.T. Andrews showed that a bounded subset A of a Banach space X is a Dunford–Pettis set if and only if $\lim_{n \rightarrow \infty} \sup_{x \in A} |\langle x_n^*, x \rangle| = 0$ for every weakly null sequence $(x_n^*)_n$ in X^* . Inspired by this fact, we introduce the notion of p -(V) sets and prove that a Banach space X has the DPP_p if and only if each relatively weakly compact subset of X^* is a p -(V) set. We call these sets p -(V) sets because 1-(V) sets are precisely (V) sets (see [25] for the definition of (V) sets).

Section 4 is inspired by the work of M. Kačena, O.F.K. Kalenda and J. Spurný on quantification of Dunford–Pettis property in [21]. In [21], the authors use the Mackey topology to define two quantities $\tilde{c}a_\rho(\cdot)$ and $\tilde{c}a_{\rho^*}(\cdot)$, which are utilized to quantify the Dunford–Pettis property. In this section, we introduce a complete locally convex topology ρ_p^* by relatively weakly p -compact sets and define a quantity $\tilde{c}a_{\rho_p^*}(\cdot)$. We utilize this quantity to quantify Dunford–Pettis property of order p .

The reader is referred to [17] and [24] for any unexplained notation or terminology.

2. p -Converging operators

Definition 2.1. [4] Let $1 \leq p \leq \infty$. An operator $T : X \rightarrow Y$ is said to be *p -converging* if T transforms weakly p -summable sequences of X (weakly null sequences of X for $p = \infty$) into norm null sequences of Y . The ideal of p -converging operators is denoted by \mathcal{C}_p .

It is mentioned in [9] that \mathcal{C}_p forms a closed, injective and non-surjective operator ideal. The following characterization of p -converging operators, which appears as a remark before [9, Proposition 1.6], is simple, but useful.

Lemma 2.2. [9] *Let $1 \leq p < \infty$. The following are equivalent for an operator $T : X \rightarrow Y$:*

- (1) T is p -converging;
- (2) TS is compact for any operator $S \in \mathcal{L}(l_{p^*}, X)$ ($\mathcal{L}(c_0, X)$ for $p = 1$).

Before another frequently useful characterization of p -converging operators is given, we recall the notion of weakly p -convergent sequences introduced in [11]. A sequence $(x_n)_n$ in a Banach space X is said to be *weakly p -convergent to $x \in X$* ($1 \leq p \leq \infty$) if the sequence $(x_n - x)_n$ is weakly p -summable in X . Weakly ∞ -convergent sequences are simply the weakly convergent sequences. Obviously, if $1 \leq p_1 < p_2 \leq \infty$, then a sequence $(x_n)_n$ is weakly p_2 -convergent whenever $(x_n)_n$ is weakly p_1 -convergent. It is natural to generalize weakly Cauchy sequences to the general case $1 \leq p \leq \infty$.

Definition 2.3. Let $1 \leq p \leq \infty$. We say that a sequence $(x_n)_n$ in a Banach space X is *weakly p -Cauchy* if for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(x_{k_n} - x_{j_n})_n$ is weakly p -summable in X .

Obviously, every weakly p -convergent sequence is weakly p -Cauchy, and the weakly ∞ -Cauchy sequences are precisely the weakly Cauchy sequences.

Theorem 2.4. *Let $1 \leq p \leq \infty$. An operator $T : X \rightarrow Y$ is p -converging if and only if T sends weakly p -Cauchy sequences onto norm convergent sequences.*

Proof. Suppose that $T : X \rightarrow Y$ is p -converging. Let $(x_n)_n$ be a weakly p -Cauchy sequence in X . Then, for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(Tx_{k_n} - Tx_{j_n})_n$ converges to 0 in norm and hence $(Tx_n)_n$ converges in norm.

Conversely, suppose that $(x_n)_n$ is a weakly p -summable sequence in X . Then $(x_n)_n$ is weakly p -Cauchy. By the assumption, $(Tx_n)_n$ converges in norm. Since $(x_n)_n$ is weakly null, $(Tx_n)_n$ is weakly null and hence norm null. This completes the proof. \square

Following [28], recall that a subset K of a Banach space X is *relatively weakly p -compact* ($1 \leq p < \infty$) if there is a sequence $(x_n)_n \in l_p^w(X)$ such that $K \subset E_{(x_n)_n}(B_{l_{p^*}})$ for $1 < p < \infty$ ($E_{(x_n)_n}(B_{c_0})$ for $p = 1$), where $E : l_p^w(X) \rightarrow \mathcal{L}(l_{p^*}, X)$ is the isometric isomorphism defined by $(x_n)_n \mapsto E_{(x_n)_n}, E_{(x_n)_n}((\alpha_n)_n) = \sum_n \alpha_n x_n$. A subset K of a Banach space X is said to be *relatively weakly p -precompact* if every sequence in K admits a weakly p -convergent subsequence (see [8]). Bessaga–Pełczyński Selection Principle yields that every relatively weakly p -compact set is relatively weakly p -precompact for any $1 < p < \infty$. Indeed, suppose that K is relatively weakly p -compact and $1 < p < \infty$. Let S be an operator from l_{p^*} to X such that $K \subset S(B_{l_{p^*}})$. Fix a sequence $(x_n)_n$ in K and find a sequence $(z_n)_n$ in $B_{l_{p^*}}$ such that $x_n = Sz_n$ for each n . By passing to subsequences, we may assume that $(z_n)_n$ is weakly convergent to some $z \in B_{l_{p^*}}$. If $(z_n)_n$ is norm convergent to z , then $(x_n)_n$ is norm convergent and hence admits a weakly p -convergent subsequence. If $(z_n)_n$ is not norm convergent to z , Bessaga–Pełczyński Selection Principle yields a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that $(z_{n_k} - z)_k$ is equivalent to the unit vector basis of l_{p^*} . Hence $(z_{n_k})_k$ is weakly p -convergent and so the same is true for $(x_{n_k})_k$. But the converse needs not to be true. Let $X = (\sum_{n=1}^{\infty} l_1^n)_{p^*}$ ($1 < p < \infty$). Then, as the same argument (the classical gliding hump argument) for $B_{l_{p^*}}$, it is easy to show that B_X is relatively weakly p -precompact. If B_X is relatively weakly p -compact, then X is isomorphic to a complemented subspace

of a quotient of l_{p^*} and hence X is isomorphic to a quotient of l_{p^*} . This implies that $X^* = (\sum_{n=1}^{\infty} l_{\infty}^n)_p$ is isomorphic to a subspace of l_p . This is impossible because the Maurey–Pisier theorem means that the space $(\sum_{n=1}^{\infty} l_{\infty}^n)_p$ does not have finite cotype.

Another counterexample is $L_p(1 < p < \infty, p \neq 2)$. For each $1 < p < \infty, p \neq 2$, B_{L_p} is relatively weakly r -precompact ([9, Proposition 1.8]), where $r = \max(p^*, 2)$, but is not relatively weakly r -compact because such L_p is not isomorphic to a quotient of l_{r^*} . Suppose that L_p is isomorphic to a quotient of l_{r^*} . For $1 < p < 2$, then L_{p^*} is isomorphic to a subspace of l_{p^*} and hence l_2 is isomorphic to a subspace of l_{p^*} , which is impossible; For $2 < p < \infty$, then L_{p^*} is isomorphic to a subspace of l_2 and hence l_{p^*} is isomorphic to a subspace of l_2 , which is false.

It should be mentioned that not every relatively weakly 1-compact set is necessarily relatively weakly 1-precompact. By definition, B_{c_0} is relatively weakly 1-compact, but is not relatively weakly 1-precompact, since the summing basis $(e_1 + \dots + e_n)_n$ of c_0 has no weakly 1-convergent subsequences.

By using the weakly p -Cauchy sequences, we can correspondingly define the conditionally weakly p -compact sets as follows:

Definition 2.5. Let $1 \leq p \leq \infty$. We say that a subset K of a Banach space X is *conditionally weakly p -compact* if every sequence in K admits a weakly p -Cauchy subsequence.

The following result, which follows from Theorem 2.4, says that p -converging operators are precisely those operators that send conditionally weakly p -compact subsets onto relatively norm compact subsets.

Theorem 2.6. Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p < \infty$. The following statements are equivalent:

- (1) T is p -converging;
- (2) T maps relatively weakly p -precompact subsets onto relatively norm compact subsets;
- (3) T maps conditionally weakly p -compact subsets onto relatively norm compact subsets;
- (4) T maps relatively weakly p -compact subsets onto relatively norm compact subsets.

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (1) are obvious. The implication (2) \Rightarrow (3) follows from Theorem 2.4. The equivalence of (1) and (4) follows from Lemma 2.2. \square

The following result shows that an operator is p -converging not precisely when its second adjoint is.

Theorem 2.7. There exists an operator T which is p -converging for each $1 \leq p \leq \infty$ such that T^{**} is not p -converging for all $1 \leq p \leq \infty$.

Proof. J. Bourgain and F. Delbaen (see [3, Theorem 4.4]) constructed a separable \mathcal{L}_{∞} -space X_{BD} such that X_{BD} has the Schur property and every infinite-dimensional closed subspace of X_{BD} contains a subspace isomorphic to l_1 . It follows from [3, Theorem 2.3] that X_{BD}^* is isomorphic to $C[0, 1]^*$. Hence X_{BD}^{**} is isomorphically universal for separable Banach spaces. Since X_{BD} has the Schur property, every operator from $l_p(1 < p < \infty)$ and from c_0 into X_{BD} is compact. By Lemma 2.2, every operator with domain X_{BD} is p -converging for each $1 \leq p < \infty$. In particular, $I_{X_{BD}}$ is p -converging. But since X_{BD}^{**} is isomorphically universal for separable Banach spaces, there exists a closed subspace $X_{p^*}(X_0$ for $p = 1)$ of X_{BD}^{**} such that X_{p^*} is isomorphic to l_{p^*} for $1 < p < \infty$ (X_0 is isomorphic to c_0 for $p = 1$). This implies that $I_{X_{BD}^{**}} = I_{X_{p^*}^{**}}$ is not l_{p^*} -strictly singular for $1 < p < \infty$ (c_0 -strictly singular for $p = 1$). Thus $I_{X_{BD}^{**}} = I_{X_{p^*}^{**}}$ is not p -converging. For $p = \infty$, $I_{X_{BD}}$ is obviously completely continuous, but $I_{X_{BD}^{**}} = I_{X_{BD}^{**}}$ is not completely continuous because X_{BD}^{**} has not the Schur property. \square

3. Dunford–Pettis property of order p

J.M.F. Castillo and F. Sánchez extended the classical *DPP* to the general case for $1 \leq p \leq \infty$ in [9]. Let $1 \leq p \leq \infty$. A Banach space X is said to have the *Dunford–Pettis property of order p* (in short, DPP_p) if for every Banach space Y , every weakly compact operator $T : X \rightarrow Y$ is p -converging. Many classical spaces failing the *DPP* enjoy the DPP_p . A simple observation is that if a Banach space X has cotype $q < \infty$, then X has the DPP_p for any $1 < p < q^*$. Thus, the classical Hardy space H^1 , which fails the *DPP* (see [15, p. 45]), has the DPP_p for any $1 < p < 2$. It is known that all the Lorentz function spaces $\Lambda(W, 1)$'s fail the *DPP* (see [15, p. 45]). But there are certain positive results for DPP_p . For example, if we take $W(t) = \frac{1}{2\sqrt{t}}, t \in (0, 1]$, then the space $\Lambda(W, 1)$ has the DPP_p for some $1 < p \leq 2$. Another non-reflexive space failing the *DPP* is the interesting space L built in [23]. Indeed, it was shown in [2, Theorem 14] that even duals of L fail the *DPP* and odd duals of L fail the surjective *DPP*, which is genuinely weaker than the *DPP*. Moreover, F. Bombal, P. Cembranos and J. Mendoza proved that for any $1 \leq p < \infty$, every operator from L into l_p is compact (see [2, Theorem 14]). This means that L^* has the DPP_p for any $1 < p < \infty$. More examples can be found in [9].

The following result is an internal characterization of the DPP_p . It is a refinement of [9, Proposition 3.2].

Theorem 3.1. *Let $1 < p < \infty$ and X be a Banach space. The following are equivalent:*

- (1) X has the DPP_p ;
- (2) Every weakly compact operator T from X into c_0 is p -converging;
- (3) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every weakly p -Cauchy sequence $(x_n)_n$ in X and every weakly null sequence $(x_n^*)_n$ in X^* ;
- (4) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n)_n \in l_p^w(X)$ and every weakly null sequence $(x_n^*)_n$ in X^* ;
- (5) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n)_n \in l_p^w(X)$ and every weakly Cauchy sequence $(x_n^*)_n$ in X^* .

Proof. The implication (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (3). Assume that we have a weakly p -Cauchy sequence $(x_n)_n$ in X and a weakly null sequence $(x_n^*)_n$ in X^* . Define an operator $T : X \rightarrow c_0$ by $Tx = (\langle x_n^*, x \rangle)_n$. Since $(x_n^*)_n$ converges to 0 weakly, T^* is weakly compact and so is T . By (2), T is p -converging. By Theorem 2.4, $(Tx_n)_n$ converges to some $\xi = (\xi_k)_k \in c_0$ in norm. Let $\epsilon > 0$. There exists a positive integer N_1 such that $\|Tx_n - \xi\| < \frac{\epsilon}{2}$ for all $n > N_1$. Choose another positive integer N_2 such that $|\xi_k| < \frac{\epsilon}{2}$ for all $k > N_2$. By the definition of T , we have $|\langle x_n^*, x_n \rangle| < \epsilon$ for all $n > \max(N_1, N_2)$. Thus $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$.

(3) \Rightarrow (4) is trivial.

(4) \Rightarrow (5). Suppose that $(x_n)_n$ is weakly p -summable in X and $(x_n^*)_n$ is weakly Cauchy in X^* , yet $(\langle x_n^*, x_n \rangle)_n$ does not converge to 0. By passing to subsequences, we may assume that $|\langle x_n^*, x_n \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and all $n \in \mathbb{N}$. Since $(x_n)_n$ is weakly p -summable and in particular weakly null, there exists a subsequence $(x_{k_n})_n$ of $(x_n)_n$ such that $|\langle x_n^*, x_{k_n} \rangle| < \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Since $(x_n^*)_n$ is weakly Cauchy, we see that $(x_{k_n}^* - x_n^*)_n$ is weakly null. By (4), $\lim_{n \rightarrow \infty} \langle x_{k_n}^* - x_n^*, x_{k_n} \rangle = 0$. This implies that $|\langle x_{k_n}^* - x_n^*, x_{k_n} \rangle| < \frac{\epsilon_0}{3}$ for n large enough. But for such n 's, we have

$$\epsilon_0 < |\langle x_{k_n}^*, x_{k_n} \rangle| \leq |\langle x_{k_n}^* - x_n^*, x_{k_n} \rangle| + |\langle x_n^*, x_{k_n} \rangle| < \frac{5\epsilon_0}{6}.$$

(5) \Rightarrow (1). Let $T : X \rightarrow Y$ be a weakly compact operator. Let us suppose that T is not p -converging. Then we obtain a weakly p -summable sequence $(x_n)_n$ in X and $\epsilon_0 > 0$ such that $\|Tx_n\| > \epsilon_0 (n = 1, 2, \dots)$. Pick $y_n^* \in Y^*$ such that $\langle y_n^*, Tx_n \rangle = \|Tx_n\|$ and $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$. Since T is weakly compact, so is T^* . Hence there is a subsequence $(y_{k_n}^*)_n$ of $(y_n^*)_n$ such that the sequence $(T^*y_{k_n}^*)_n$ converges weakly and hence

is weakly Cauchy. The assumption ensures that the sequence $(\langle T^*y_{k_n}^*, x_{k_n} \rangle)_n = (\|Tx_{k_n}\|)_n$ converges to 0, which is a contradiction. \square

It is worth mentioning that the equivalence (1) \Leftrightarrow (2) of [Theorem 3.1](#) is an analogue of the same characterization of *DPP*. There is another known characterization of *DPP*: a Banach space X has the *DPP* if and only if, for all Banach spaces Y , every weakly compact operator $T : Y \rightarrow X$ has a completely continuous adjoint if and only if the same holds for $Y = l_1$. A natural question arises: Does the analogue hold for DPP_p ? Before we answer this question, we prove the following result.

Corollary 3.2. *Let X be a Banach space containing no copy of l_1 and $1 < p < \infty$. The following statements are equivalent:*

- (1) X^* has the DPP_p ;
- (2) For all Banach spaces Y , every weakly compact operator $T : Y \rightarrow X$ has the p -converging adjoint;
- (3) Every weakly compact operator $T : l_1 \rightarrow X$ has the p -converging adjoint;
- (4) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n^*)_n \in l_p^w(X^*)$ and every weakly Cauchy sequence $(x_n)_n$ in X ;
- (5) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every weakly p -Cauchy sequence $(x_n^*)_n$ in X^* and every weakly null sequence $(x_n)_n$ in X ;
- (6) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n^*)_n \in l_p^w(X^*)$ and every weakly null sequence $(x_n)_n$ in X .

Proof. The implication (1) \Rightarrow (2) follows from the definition of DPP_p . The implication (2) \Rightarrow (3) is obvious.

(3) \Rightarrow (4). Assuming the contrary, we can find $(x_n^*)_n \in l_p^w(X^*)$ and a weakly Cauchy sequence $(x_n)_n$ in X such that $|\langle x_n^*, x_n \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and all $n \in \mathbb{N}$. Since $(x_n^*)_n$ is weakly null, there exists a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ such that $|\langle x_{k_n}^*, x_n \rangle| < \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Thus $|\langle x_{k_n}^*, x_n - x_{k_n} \rangle| > \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Define an operator $S : X^* \rightarrow c_0$ by

$$Sx^* = (\langle x^*, x_n - x_{k_n} \rangle)_n, \quad x^* \in X^*.$$

It is easy to check that $S^*e_n = x_n - x_{k_n}$ ($n = 1, 2, \dots$), where $(e_n)_n$ is the unit vector basis of l_1 . Thus the operator S^* maps l_1 into X and is weakly compact. By (3), the operator S^{**} is p -converging. Thus $\lim_{n \rightarrow \infty} \|Sx_{k_n}^*\| = \lim_{n \rightarrow \infty} \|S^{**}x_{k_n}^*\| = 0$. It follows from the definition of the operator S that $\lim_{n \rightarrow \infty} |\langle x_{k_n}^*, x_n - x_{k_n} \rangle| = 0$, which is a contradiction.

The proof of (4) \Rightarrow (5) is similar to [Theorem 3.1](#) (4) \Rightarrow (5).

(5) \Rightarrow (6) is trivial.

(6) \Rightarrow (1). By [Theorem 3.1](#), it is enough to verify that for every $(x_n^*)_n \in l_p^w(X^*)$ and every weakly null sequence $(x_n^{**})_n$ in X^{**} , the sequence $(\langle x_n^{**}, x_n^* \rangle)_n$ converges to 0. Now we suppose that it is false. Then, by passing to subsequences, we may assume that $|\langle x_n^{**}, x_n^* \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and all $n \in \mathbb{N}$. Of course, we may also assume that $\|x_n^{**}\| \leq 1$ for all $n \in \mathbb{N}$. It follows from Goldstine’s Theorem that for each $n \in \mathbb{N}$, there exists an $x_n \in B_X$ such that $|\langle x_n - x_n^{**}, x_n^* \rangle| < \frac{\epsilon_0}{2}$. Then $|\langle x_n^*, x_n \rangle| > \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. By Rosenthal’s Theorem, $(x_n)_n$ has a weakly Cauchy subsequence, which is still denoted by $(x_n)_n$. Then there exists a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ such that $|\langle x_{k_n}^*, x_n \rangle| < \frac{\epsilon_0}{3}$ for all $n \in \mathbb{N}$. By (6), we get $\lim_{n \rightarrow \infty} \langle x_{k_n}^*, x_{k_n} - x_n \rangle = 0$, which implies that $|\langle x_{k_n}^*, x_{k_n} - x_n \rangle| < \frac{\epsilon_0}{6}$ for n large enough. It is easy to verify that for such n ’s, $|\langle x_{k_n}^*, x_{k_n} \rangle| < \frac{\epsilon_0}{2}$. This contradiction completes the proof. \square

By [Corollary 3.2](#)(1), (2) and (3), we see that the above mentioned question is equivalent to the following statement: a Banach space X containing no copy of l_1 has the DPP_p if and only if X^* has the DPP_p . However, this is not true in general. In fact, it is shown in [9, Example 3.3] that Tsirelson’s space T has the DPP_p for all $p < \infty$, but T^* does not have the DPP_p for any $p > 1$.

Definition 3.3. Let $1 < p < \infty$. We say that a Banach space X has the *hereditary Dunford–Pettis property of order p* (in short, hereditary DPP_p) if every closed subspace of X has the DPP_p .

Lemma 3.4. [16] If $(x_n)_n$ is a normalized weakly null sequence of a space X such that no subsequence of it is equivalent to the unit vector basis $(e_n)_n$ of c_0 , then $(x_n)_n$ has a subsequence $(y_n)_n$ for which given any subsequence $(z_n)_n$ of $(y_n)_n$ and any sequence $(\alpha_n)_n \notin c_0$ we have $\sup_n \left\| \sum_{k=1}^n \alpha_k z_k \right\| = +\infty$.

Theorem 3.5. Let X be a Banach space and $1 < p < \infty$. The following are equivalent:

- (1) X has the hereditary DPP_p ;
- (2) Every normalized weakly p -summable sequence in X admits a subsequence that is equivalent to the unit vector basis of c_0 ;
- (3) Every weakly p -summable sequence in X admits a weakly 1-summable subsequence;
- (4) Every weakly p -summable sequence in X admits a subsequence $(y_n)_n$ such that $\sup_N \left\| \sum_{n=1}^N y_n \right\| < \infty$.

Proof. (1) \Rightarrow (2). Let $(x_n)_n$ be a normalized weakly p -summable sequence in X such that it admits no subsequence that is equivalent to the unit vector basis $(e_n)_n$ of c_0 . It follows from Lemma 3.4 that $(x_n)_n$ has a subsequence $(y_n)_n$ as stated in Lemma 3.4. By Bessaga–Pelczyński Selection Principle, we may assume that $(y_n)_n$ is a basic sequence. Let $X_0 = \overline{\text{span}}\{y_n : n = 1, 2, \dots\}$. Let $(y_n^*)_n \subset X_0^*$ be the coefficient functionals of the basic sequence $(y_n)_n$. For each N , define a projection $P_N : X_0 \rightarrow X_0$ by

$$P_N(y) = \sum_{n=1}^N \langle y_n^*, y \rangle y_n, \quad y \in X_0.$$

Then the projection P_N 's are uniformly bounded in operator norm. An easy verification shows that $P_N^{**} y^{**} = \sum_{n=1}^N \langle y^{**}, y_n^* \rangle y_n$ for all $y^{**} \in X_0^{**}$. Lemma 3.4 and the uniform boundedness of the projection P_N 's imply that $(\langle y^{**}, y_n^* \rangle)_n \in c_0$ for all $y^{**} \in X_0^{**}$, that is, $(y_n^*)_n$ is weakly null. Since $\langle y_n^*, y_n \rangle = 1$ for all $n \in \mathbb{N}$, it follows from Theorem 3.1 again that X_0 fails the DPP_p .

(2) \Rightarrow (3) and (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (1). Take a subspace X_0 of X that fails the DPP_p . Appealing to Theorem 3.1, we obtain a weakly compact operator $T : X_0 \rightarrow c_0$ which is not p -converging. Then there is a normalized weakly p -summable sequence $(x_n)_n$ in X such that $\|Tx_n\| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Bessaga–Pelczyński Selection Principle allows us to assume that the sequence $(Tx_n)_n$ is equivalent to the unit vector basis $(e_n)_n$ of c_0 . By the weak compactness of T , the sequence $(x_n)_n$ admits no subsequence equivalent to the unit vector basis $(e_n)_n$. By Lemma 3.4, the sequence $(x_n)_n$ admits a subsequence $(y_n)_n$ for which given any subsequence $(z_n)_n$ of $(y_n)_n$, one has $\sup_N \left\| \sum_{n=1}^N z_n \right\| = \infty$. \square

Remark 3.6.

- (a) The equivalence of (1) and (2) in Theorem 3.5 is the p -version of [12, Proposition 2];
- (b) In [10], J.M.F. Castillo and F. Sánchez say that a Banach space X has property weak W_1 if every weakly null sequence admits a weakly 1-summable subsequence. The authors observed that property weak W_1 is equivalent to the hereditary Dunford–Pettis property. The equivalence of (1) and (3) in Theorem 3.5 can be considered to be the p -version of their observation;
- (c) The equivalence of (1) and (4) in Theorem 3.5 is the p -version of [6, Lemma].

Definition 3.7. Let $1 < p \leq \infty$. We say that a Banach space X has the *weak Banach–Saks property of order p* if each weakly p -summable sequence in X admits a subsequence $(x_n)_n$ such that $\lim_{n \rightarrow \infty} \left\| \sum_{k=1}^n x_k \right\| / n^{\frac{1}{p}} = 0$.

For $p = \infty$, the weak Banach–Saks property of order p is precisely the weak Banach–Saks property.

A direct consequence of [Theorem 3.5](#) is the following corollary which is the p -version of [\[15, Theorem 6\]](#).

Corollary 3.8. *If a Banach space X has the hereditary DPP_p , then X has the weak Banach–Saks property of order p .*

Definition 3.9. Let X be a Banach space and $1 \leq p < \infty$. We say that a bounded subset K of X^* is a p –(V) set if $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$ for every $(x_n)_n \in l_p^w(X)$.

1–(V) sets are (V) sets. R. Cilia and G. Emmanuele proved that a bounded subset K of X^* is a 1–(V) set if and only if for every operator T from c_0 into X , the subset $T^*(K)$ is relatively norm compact ([\[13, Proposition 5\]](#)). J. M. F. Castillo and F. Sánchez said that a Banach space $X \in W_p$ ($1 \leq p < \infty$) if any bounded sequence in X admits a weakly p -convergent subsequence (see [\[11, Definition 2.1\]](#)). We use this notion to characterize p –(V) sets and generalize the result of R. Cilia and G. Emmanuele to $1 < p < \infty$.

Theorem 3.10. *Let $1 < p < \infty$ and X be a Banach space. The following statements are equivalent about a bounded subset K of X^* :*

- (1) K is a p –(V) set;
- (2) For all spaces $Y \in W_p$ and for every operator T from Y into X , the subset $T^*(K)$ is relatively norm compact;
- (3) For every operator T from l_{p^*} into X , the subset $T^*(K)$ is relatively norm compact.

Proof. (1) \Rightarrow (2). Let T be an operator from $Y \in W_p$ into X such that $T^*(K)$ is not relatively norm compact. Then there exists a sequence $(x_n^*)_n$ in K such that $(T^*x_n^*)_n$ admits no norm convergent subsequences. Since Y^* is reflexive, by passing to a subsequence if necessary we may assume that $(T^*x_n^*)_n$ converges weakly to some $y^* \in Y^*$ and $\|T^*x_n^* - y^*\| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $n \in \mathbb{N}$. For each n , choose y_n with $\|y_n\| \leq 1$ such that $|\langle T^*x_n^* - y^*, y_n \rangle| > \epsilon_0$. Since $Y \in W_p$, by passing to a subsequence again if necessary one can assume that the sequence $(y_n)_n$ is weakly p -convergent to some $y \in Y$. Thus, by hypothesis, we get $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, Ty_n - Ty \rangle| = 0$. Note that, for each $n \in \mathbb{N}$,

$$|\langle T^*x_n^* - y^*, y_n \rangle| \leq |\langle x_n^*, Ty_n - Ty \rangle| + |\langle x_n^*, Ty \rangle - \langle y^*, y \rangle| + |\langle y^*, y - y_n \rangle|.$$

This implies that $\lim_{n \rightarrow \infty} \langle T^*x_n^* - y^*, y_n \rangle = 0$, which is a contradiction.

(2) \Rightarrow (3) is immediate because $l_{p^*} \in W_p$.

(3) \Rightarrow (1). Let $(x_n)_n \in l_p^w(X)$. Then there exists an operator T from l_{p^*} into X such that $Te_n = x_n$ for all $n \in \mathbb{N}$. It follows from (3) that $T^*(K)$ is relatively norm compact. By the well-known characterization of relatively norm compact subsets of l_p (see [\[16, p. 6\]](#)), one can derive that $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$. \square

Theorem 3.11. *Let $1 < p < \infty$. A Banach space X has the DPP_p if and only if each relatively weakly compact subset of X^* is a p –(V) set.*

Proof. The sufficient part follows directly from the definitions of the DPP_p and the p –(V) sets. On the other hand, let K be a relatively weakly compact subset of X^* . By the Davis–Figiel–Johnson–Pełczyński

factorization lemma ([14, §2. Lemma 1]), there exists a reflexive space Z , which is a linear subspace of X^* , such that the inclusion map $J : Z \rightarrow X^*$ is bounded and B_Z contains K . Since Z is reflexive, there is an operator $T : X \rightarrow Z^*$ such that $T^* = J$. By the assumption, T is p -converging. Thus the set $T^*(B_Z) = J(B_Z) = B_Z$ is a p -(V) set. Thus K is a p -(V) set. \square

4. Quantifying Dunford–Pettis property of order p

Let us fix some notations. If A and B are nonempty subsets of a Banach space X , we set

$$d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\},$$

$$\widehat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

Thus, $d(A, B)$ is the ordinary distance between A and B , and $\widehat{d}(A, B)$ is the non-symmetrized Hausdorff distance from A to B .

Let A be a bounded subset of a Banach space X . The Hausdorff measure of non-compactness of A is defined by

$$\chi(A) = \inf\{\widehat{d}(A, F) : F \subset X \text{ finite}\},$$

$$\chi_0(A) = \inf\{\widehat{d}(A, F) : F \subset A \text{ finite}\}.$$

Then $\chi(A) = \chi_0(A) = 0$ if and only if A is relatively norm compact. It is easy to verify that

$$\chi(A) \leq \chi_0(A) \leq 2\chi(A). \quad (4.1)$$

The de Blasi measure of weak non-compactness of A is defined by

$$\omega(A) = \inf\{\widehat{d}(A, K) : \emptyset \neq K \subset X \text{ is weakly compact}\}.$$

Then $\omega(A) = 0$ if and only if A is relatively weakly compact.

Let X be a Banach space and let \mathcal{F} be the family of all weakly compact subsets of B_{X^*} . For $F \in \mathcal{F}$, define a semi-norm q_F on X^{**} by

$$q_F(x^{**}) = \sup_{x^* \in F} |\langle x^{**}, x^* \rangle|, \quad x^{**} \in X^{**}.$$

The locally convex topology generated by the family of semi-norms $\{q_F : F \in \mathcal{F}\}$ is called the Mackey topology, denoted by $\tau(X^{**}, X^*)$. The restriction to X of the Mackey topology $\tau(X^{**}, X^*)$ is called the Right topology in [26]. This topology is denoted by ρ_X or simply ρ when X is obvious.

In this section, we introduce a new locally convex topology. Let X be a Banach space and let $1 \leq p < \infty$. Let \mathcal{F}_p be the family of all relatively weakly p -compact subsets of X . For $F \in \mathcal{F}_p$, we define a semi-norm q_F on X^* by

$$q_F(x^*) = \sup_{x \in F} |\langle x^*, x \rangle|, \quad x^* \in X^*.$$

The locally convex topology generated by the family of semi-norms $\{q_F : F \in \mathcal{F}_p\}$ is denoted by ρ_p^* when X is obvious. Applying Grothendieck's Completeness Theorem ([27, p. 148]), we obtain that the space (X^*, ρ_p^*) is complete. Indeed, let f be a linear functional on X such that $f|_F$ is $\|\cdot\|$ -continuous on each $F \in \mathcal{F}_p$. Let $(x_n)_n$ be a norm null sequence in X . Find a subsequence $(x_{k_n})_n$ of $(x_n)_n$ such that $\sum_n \|x_{k_n}\|^p < \infty$. Let

$F = (x_{k_n})_n \cup \{0\}$. Then $\lim_{n \rightarrow \infty} \|f(x_{k_n})\| = 0$. This implies that $f \in X^*$. By Grothendieck’s Completeness Theorem, the locally convex space (X^*, ρ_p^*) is complete.

Hence, a bounded subset A of X^* is relatively ρ_p^* -compact if and only if A is totally bounded, equivalently, the set $A|_F = \{x^*|_F : x^* \in A\}$ is totally bounded in $l_\infty(F)$ for each relatively weakly p -compact subset $F \subset B_X$. So, if we set

$$\chi_m^p(A) = \sup\{\chi_0(A|_F) : F \in \mathcal{F}_p, F \subset B_X\},$$

then A is relatively ρ_p^* -compact if and only if $\chi_m^p(A) = 0$.

Let $(x_n^*)_n$ be a bounded sequence in X^* . We set

$$ca_{\rho_p^*}((x_n^*)_n) = \sup_{F \in \mathcal{F}_p, F \subset B_X} \inf_n \sup\{q_F(x_k^* - x_l^*) : k, l \geq n\},$$

and

$$\tilde{ca}_{\rho_p^*}((x_n^*)_n) = \inf\{ca_{\rho_p^*}((x_{k_n}^*)_n) : (x_{k_n}^*)_n \text{ is a subsequence of } (x_n^*)_n\}.$$

The following result contains two topological characterizations of DPP_p .

Theorem 4.1. *The following are equivalent about a Banach space X and $1 < p < \infty$:*

- (1) X has the DPP_p ;
- (2) Every weakly p -summable sequence in X is p -null;
- (3) Every weakly convergent sequence in X^* is ρ_p^* -convergent.

Proof. The equivalence of (1) and (2) is essentially [Theorem 3.11](#). The implication (3) \Rightarrow (1) follows from [Theorem 3.1](#). It remains to prove (1) \Rightarrow (3).

Let $(x_n^*)_n$ be weakly null in X^* . Define an operator $T : X \rightarrow c_0$ by

$$Tx = (\langle x_n^*, x \rangle)_n, \quad x \in X.$$

Since $(x_n^*)_n$ is weakly null, T is weakly compact (see [\[16, p. 114\]](#)). By (1), we get T is p -converging. Let $F \in \mathcal{F}_p$. It follows from [Theorem 2.6](#) that TF is relatively norm compact in c_0 . By the well-known characterization of relatively norm compact subsets of c_0 (see [\[16, p. 15\]](#)), we get

$$\lim_{n \rightarrow \infty} q_F(x_n^*) = \lim_{n \rightarrow \infty} \sup_{x \in F} |\langle x_n^*, x \rangle| = 0,$$

which implies that $(x_n^*)_n$ is ρ_p^* -null. \square

Let X be a Banach space and A be a bounded subset of X^* . For $1 \leq p < \infty$, we set

$$\iota_p(A) = \sup\{\limsup_n \sup_{x^* \in A} |\langle x^*, x_n \rangle| : (x_n)_n \in l_p^w(X), (x_n)_n \subset B_X\},$$

Obviously, $\iota_p(A) = 0$ if and only if A is a p -(V) set.

Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p < \infty$. We set

$$uc_p^1(T) = \sup\{\limsup_n \|Tx_n\| : (x_n)_n \in l_p^w(X), (x_n)_n \subset B_X\}.$$

The following lemma says, in particular, that p -(V) sets coincide with relatively ρ_p^* -compact sets. Its proof is similar to [\[22, Lemma 5.6\]](#).

Lemma 4.2. *Let X be a Banach space, $1 \leq p < \infty$ and A be a bounded subset of X^* . Then*

$$\frac{1}{8} \chi_m^p(A) \leq \iota_p(A) \leq \chi_m^p(A).$$

The following theorem is a p -analogue to [21, Theorem 5.2].

Theorem 4.3. *Let X be a Banach space and $1 < p < \infty$. The following are equivalent:*

- (1) X has the DPP_p ;
- (2) $uc_p^1(T) \leq \omega(T^*)$ for every operator T from X into any Banach space Y ;
- (3) $\iota_p(A) \leq \omega(A)$ for every bounded subset A of X^* ;
- (4) $\tilde{ca}_{\rho_p^*}((x_n^*)_n) \leq 2\omega((x_n^*)_n)$ whenever $(x_n^*)_n$ is a bounded sequence in X^* ;
- (5) $\chi_m^p(A) \leq 2\omega(A)$ for every bounded subset A of X^* .

Proof. (2) \Rightarrow (1) is obvious. (3) \Rightarrow (1) follows from Theorem 3.11. (5) \Rightarrow (1) follows from Theorem 3.11 and Lemma 4.2.

(1) \Rightarrow (2). Let Y be a Banach space and let $T \in \mathcal{L}(X, Y)$. Let $\epsilon > 0$ be such that $T^*B_{Y^*} \subset K + \epsilon B_{X^*}$, $K \subset X^*$ is weakly compact. Let $(x_n)_n \in l_p^w(X)$ and $(x_n)_n \subset B_X$. Since X has the DPP_p , it follows from Theorem 3.11 that $\limsup_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$. Let $c > 0$. Then there exists a positive integer N such that $\sup_{x^* \in K} |\langle x^*, x_n \rangle| < c$ for each $n \geq N$. For each $n \in \mathbb{N}$, pick $y_n^* \in B_{Y^*}$ with $\|Tx_n\| = \langle y_n^*, Tx_n \rangle$. Since $T^*B_{Y^*} \subset K + \epsilon B_{X^*}$, then, for each $n \in \mathbb{N}$, there exists $x_n^* \in K$ such that $\|T^*y_n^* - x_n^*\| \leq \epsilon$. Then, for $n \geq N$, we get

$$\begin{aligned} \|Tx_n\| &= \langle T^*y_n^*, x_n \rangle \\ &\leq \epsilon + |\langle x_n^*, x_n \rangle| \\ &\leq \epsilon + \sup_{x^* \in K} |\langle x^*, x_n \rangle| \\ &\leq \epsilon + c. \end{aligned}$$

This yields $\limsup_n \|Tx_n\| \leq \epsilon + c$. Since $c > 0$ is arbitrary, we obtain $\limsup_n \|Tx_n\| \leq \epsilon$ and hence $uc_p^1(T) \leq \epsilon$.

This proves $uc_p^1(T) \leq \omega(T^*)$.

(1) \Rightarrow (3). Let A be a bounded subset of X^* . Let $\epsilon > 0$ be such that $A \subset K + \epsilon B_{X^*}$, $K \subset X^*$ is weakly compact. Let $(x_n)_n$ be a weakly p -summable sequence in B_X . For each $x^* \in A$, there exists $z^* \in K$ such that $\|x^* - z^*\| \leq \epsilon$. This yields

$$|\langle x^*, x_n \rangle| \leq \epsilon + \sup_{x^* \in K} |\langle x^*, x_n \rangle| \quad (n = 1, 2, \dots).$$

Since X has the DPP_p , it follows from Theorem 3.11 that $\limsup_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$. Thus we get $\limsup_n \sup_{x^* \in A} |\langle x^*, x_n \rangle| \leq \epsilon$, which completes the proof (1) \Rightarrow (3).

(1) \Rightarrow (4). Let $(x_n^*)_n$ be a bounded sequence in X^* . Let $\epsilon > 0$ be such that $(x_n^*)_n \subset K + \epsilon B_{X^*}$, $K \subset X^*$ is weakly compact. For each x_n^* , there exists $z_n^* \in K$ such that $\|x_n^* - z_n^*\| \leq \epsilon$. Since K is weakly compact, there exists a weakly convergent subsequence $(z_{k_n}^*)_n$ of $(z_n^*)_n$. By Theorem 4.1, we see that the sequence $(z_{k_n}^*)_n$ is ρ_p^* -convergent and hence $ca_{\rho_p^*}((z_{k_n}^*)_n) = 0$. Note that for any $F \in \mathcal{F}_p$, $F \subset B_X$, we have

$$\begin{aligned} q_F(x_{k_i}^* - x_{k_j}^*) &\leq q_F(x_{k_i}^* - z_{k_i}^*) + q_F(z_{k_i}^* - z_{k_j}^*) + q_F(z_{k_j}^* - x_{k_j}^*) \\ &\leq 2\epsilon + q_F(z_{k_i}^* - z_{k_j}^*), i, j = 1, 2, \dots \end{aligned}$$

This yields

$$ca_{\rho_p^*}((x_{k_n}^*)_n) \leq 2\epsilon + ca_{\mathcal{F}_p}((z_{k_n}^*)_n) = 2\epsilon.$$

Hence, we get $\tilde{c}a_{\rho_p^*}((x_n^*)_n) \leq 2\epsilon$ and then $\tilde{c}a_{\rho_p^*}((x_n^*)_n) \leq 2\omega((x_n^*)_n)$.

(4) \Rightarrow (1). Let $(x_n)_n \in l_p^w(X)$ and let $(x_n^*)_n$ be weakly null in X^* . Then any subsequence of $(x_n^*)_n$ is also weakly null and hence it follows from (4) that $\tilde{c}a_{\rho_p^*}((x_{k_n}^*)_n) = 0$ for any subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$. A classical diagonal argument yields a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ which is ρ_p^* -Cauchy. Indeed, we let $f_n^{(0)} = x_n^*$. Then, by induction, we get a subsequence $(f_n^{(k)})_n$ of $(f_n^{(k-1)})_n$ with $ca_{\rho_p^*}((f_n^{(k)})_n) < \frac{1}{k}$ ($k = 1, 2, \dots$). Clearly, the diagonal sequence $(f_k^{(k)})_k$ is ρ_p^* -Cauchy. By the completeness of the topology ρ_p^* , we see that the subsequence $(x_{k_n}^*)_n$ is ρ_p^* -convergent. Since $(x_n^*)_n$ is weakly null, $(x_{k_n}^*)_n$ is ρ_p^* -null. Since $(x_n)_n$ is weakly p -summable, one has

$$|\langle x_{k_n}^*, x_{k_n} \rangle| \leq \sup_i |\langle x_{k_n}^*, x_i \rangle| \rightarrow 0 \quad (n \rightarrow \infty).$$

Then [Theorem 3.1](#) gives (1).

(1) \Rightarrow (5). Let $c > \omega(A)$. Then there exists a weakly compact subset K of X^* such that $\widehat{d}(A, K) < c$. Since X has the DPP_p , it follows from [Theorem 3.11](#) and [Lemma 4.2](#) that $\chi_m^p(K) = 0$. Let $\epsilon > 0$ and $L \in \mathcal{F}_p, L \subset B_X$. Then there exists a finite subset $F \subset K$ such that $\widehat{d}(K|_L, F|_L) < \epsilon$, so $\chi(A|_L) \leq c + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\chi(A|_L) \leq c$. By [\(4.1\)](#), we get $\chi_0(A|_L) \leq 2c$. This implies that $\chi_m^p(A) \leq 2c$, which completes the proof. \square

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