

Pełczyński's property (V^*) of order p and its quantification

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Abstract

We introduce the concepts of Pełczyński's property (V) of order p and Pełczyński's property (V^*) of order p . It is proved that, for each $1 < p < \infty$, the James p -spaces J_p enjoys Pełczyński's property (V^*) of order p and the James p^* -spaces J_{p^*} (where p^* denotes the conjugate number of p) enjoys Pełczyński's property (V) of order p . We prove that both $L_1(\mu)$ (μ a finite positive measure) and l_1 enjoy a quantitative version of Pełczyński's property (V^*).

KEYWORDSPełczyński's property (V) of order p , Pełczyński's property (V^*) of order p , quantitative Pełczyński's property (V^*) of order p **MSC (2010)**

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1 | INTRODUCTION AND NOTATIONS

Let X and Y be Banach spaces. Recall that an operator $T : X \rightarrow Y$ is called *unconditionally converging* if T takes weakly unconditionally Cauchy series in X to unconditionally converging series in Y . In his fundamental paper [40], A. Pełczyński introduced property (V). A Banach space X is said to have *Pełczyński's property (V)* if every unconditionally converging operator with domain X is weakly compact, that is, every non-weakly compact operator with domain X fixes a copy of c_0 . It is proved in [40] that X has Pełczyński's property (V) if and only if a bounded subset K of X^* is relatively weakly compact whenever

$$\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0, \text{ for each weakly unconditionally Cauchy series } \sum_{n=1}^{\infty} x_n \text{ in } X. \quad (1.1)$$

The most known classical Banach spaces that have Pełczyński's property (V) are spaces $C(\Omega)$ of continuous scalar-valued functions on compact Hausdorff space Ω (see [40]), or more generally Banach spaces whose duals are isometric to L_1 -spaces (see [30]). A. Pełczyński also asked in [40] whether for a Banach space X , the X -valued continuous functions space $C(\Omega, X)$ has Pełczyński's property (V) whenever X does. This question has been considered by several authors. P. Cembranos, N. J. Kalton, E. Saab and P. Saab [14] proved that if a Banach space X has property (u) and contains no copy of l_1 , then $C(\Omega, X)$ has Pełczyński's property (V). Property (u) and non-containment of l_1 was proved to be equivalent to hereditarily Pełczyński's property (V) by H. P. Rosenthal in [46]. N. Randrianantoanina [43] obtained a positive answer to the above question for the separable case, namely, he proved that if X is separable, then $C(\Omega, X)$ has Pełczyński's property (V) if and only if X does. He also showed that for the separable case, Pełczyński's property (V) can be lifted to the Bochner space $L_p(\mu, X)$ ($1 < p < \infty$). Pełczyński's property (V) has attracted much attention and become an important Banach space property since it was posed. Many known Banach spaces have Pełczyński's property (V). For instance, S.V. Kislyakov [35] (also F. Delbaen [19] independently) showed that the disk algebra has Pełczyński's property (V). J. Bourgain [7] proved that every non-weakly compact operator with domain H^∞ of bounded analytic functions on the unit disc D fixes a copy of l_∞ , in particular, H^∞ has Pełczyński's property

(V). H. Pfitzner [42] showed that every C^* -algebra has Pełczyński's property (V). Subsequently, C. H. Chu and P. Mellon [16] proved the larger class of JB^* -triples also have Pełczyński's property (V).

Pełczyński's property (V^*) was introduced in [40] as a dual property of Pełczyński's property (V). A Banach space X is said to have *Pełczyński's property* (V^*) if every bounded subset K of X satisfying the condition

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle x_n^*, x \rangle| = 0, \text{ for every weakly unconditionally Cauchy series } \sum_{n=1}^{\infty} x_n^* \text{ in } X^* \quad (1.2)$$

is relatively weakly compact.

We refer to [28] for a characterization of Pełczyński's property (V^*): a Banach space X has Pełczyński's property (V^*) if and only if for every Banach space Y , an operator $T : Y \rightarrow X$ is weakly compact whenever T^* is unconditionally converging. It was mentioned in [40] that if a Banach space X has Pełczyński's property (V) then X^* has Pełczyński's property (V^*) and that X has Pełczyński's property (V^*) if X^* has Pełczyński's property (V). But both of the converse implications are false. Among classical Banach spaces that have Pełczyński's property (V^*), L_1 -spaces are the most notable ones. It was generalized by E. Saab and P. Saab [47] that a Banach lattice X has Pełczyński's property (V^*) if and only if X contains no subspace isomorphic to c_0 . H. Pfitzner [41] proved that L -embedded Banach spaces, in particular non-commutative L_1 -spaces, have Pełczyński's property (V^*). In the setting of Bochner spaces, N. Randrianantoanina [44] proved that for a Banach space X , the Bochner space $L_1(\mu, X)$ has Pełczyński's property (V^*) if and only if X does. For more information and examples of spaces with Pełczyński's property (V) and Pełczyński's property (V^*), we refer to [48].

This paper deals with generalizations of Pełczyński's property (V) and Pełczyński's property (V^*). In Section 2, we introduce the concept of Pełczyński's property (V) of order p ($1 \leq p \leq \infty$) (property p -(V) in short). A bounded subset K of X^* satisfying the p -version of the condition (1.1) is called a weakly p -limited set in [15]. In the present paper, we call weakly p -limited sets p -(V) sets. 1-(V) sets are called (V)-sets in [17] and ∞ -(V) sets are called (L)-sets (see [22] for example). Property 1-(V) is precisely Pełczyński's property (V) and property ∞ -(V) is precisely the reciprocal Dunford–Pettis property (see [32] for this definition). It is clear that for each $1 < p < \infty$, a Banach space X has property p -(V) whenever X has Pełczyński's property (V). It is natural to ask whether, for each $1 < p < \infty$, there exists a space enjoying property p -(V) but failing Pełczyński's property (V). In Section 2, we show that for each $1 < p < \infty$, the James p^* -space J_{p^*} (where p^* denotes the conjugate number of p) has property p -(V) (see Theorem 2.7 below). But, J_{p^*} clearly fails Pełczyński's property (V) because J_{p^*} contains no copy of c_0 and is non-reflexive. The core of Pełczyński's property (V) is: each non-weakly compact operator fixes a copy of c_0 . A natural question arises: for each $1 < p < \infty$, does each non-weakly compact operator with domain X fix a copy of l_{p^*} whenever a Banach space X has property p -(V)? Unfortunately, this is false for property p -(V). As mentioned above, for each $1 < p < \infty$, the James p^* -space J_{p^*} has property p -(V), but the formal identity map $I_{p^*,q}$ from J_{p^*} to J_q ($q > p^*$) is non-weakly compact, strictly singular because J_{p^*} is l_{p^*} -saturated whereas J_q is l_q -saturated. It is proved in [12] that Pełczyński's property (V) is not a three-space property, that is, there exist a space X failing Pełczyński's property (V) and a closed subspace X_0 of X such that both X_0 and the quotient X/X_0 have Pełczyński's property (V). We extend this result to property p -(V) and show that property p -(V) is not a three-space property for each $1 \leq p < \infty$. In Section 2, we call a bounded subset K of X a p -(V^*) set if K satisfies the p -version of the condition (1.2). It is noted that 1-(V^*) sets are (V^*)-sets (see [40]) and ∞ -(V^*) sets are Dunford–Pettis sets (see [3]). The concept of Pełczyński's property (V^*) of order p (property p -(V^*) in short) is introduced in Section 2. Property 1-(V^*) is precisely Pełczyński's property (V^*). Similarly, for each $1 < p < \infty$, a Banach space X has property p -(V^*) whenever X has Pełczyński's property (V^*). We show that the converse is false. For each $1 < p < \infty$, the James p -space J_p fails Pełczyński's property (V^*) because J_p is not weakly sequentially complete. But J_p enjoys property p -(V^*) for each $1 < p < \infty$ as shown in Theorem 2.14 of the present paper. In [40], A. Pełczyński proved that if a Banach space X has both Pełczyński's property (V) and Pełczyński's property (V^*), then X must be reflexive. However, Theorem 2.7 and Theorem 2.14 in Section 2 tell us that the classical non-reflexive James space J has both property 2-(V) and property 2-(V^*). A. Pełczyński showed in [40] that if a Banach space X has property (V^*), then X must be weakly sequentially complete. Correspondingly, we introduce the notion of weak sequential completeness of order p and show that, for each $1 < p < 2$, a Banach space X must be weakly sequentially complete of order p whenever X has property p -(V^*).

In [6], F. Bombal studied Pełczyński's property (V^*) in vector-valued sequence spaces and proved that given a sequence $(X_n)_n$ of Banach spaces, the space $(\sum_{n=1}^{\infty} \oplus X_n)_p$ has Pełczyński's property (V^*) if and only if each X_n does. Our Theorem 3.9 complements this result. Moreover, we characterize p -(V) sets and prove that the space $(\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 < p < \infty$) has property q -(V) ($1 \leq q < \infty$) if and only if each X_n does. In particular, we show that the space $(\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 < p < \infty$ or $p = 0$) has Pełczyński's property (V) if and only if each X_n does.

Section 4 is inspired by a large number of recent results on quantitative versions of various theorems and properties of Banach spaces. For example, quantitative versions of Krein's theorem were studied in [24], quantitative versions of Eberlein–Šmulyan and Gantmacher theorems were investigated in [4], a quantitative version of James' compactness theorem in [9], quantifications of weak sequential continuity and of the Schur property in [34] and [33], quantitative Dunford–Pettis property in [31], quantification of the Banach–Saks property in [5], etc. H. Krulišová [36] introduced several possibilities of quantifying Pełczyński's property (V) . She [36] introduced a quantity $\eta(\cdot)$ measuring to what extent a bounded subset K of X^* fails to be a (V) -set and prove a quantitative version of the Pełczyński's characterization of Pełczyński's property (V) . Moreover, a quantitative version of Pełczyński's result about $C(K)$ spaces is given in [36]. More precisely, she proved that the space $C_0(\Omega)$ enjoys the quantitative property $(V_q)_\omega^*$ with constant π (2 in the real case) for every locally compact Hausdorff space Ω . Later, H. Krulišová [37] generalized this result by quantifying Pfitzner's theorem. In Section 4, we introduce the concepts of quantitative Pełczyński's property (V^*) of order p and quantitative Pełczyński's property (V) of order p . First we introduce a quantity $\iota_p(\cdot)$ ($1 \leq p < \infty$) measuring to what extent a bounded subset K of X fails to be a p - (V^*) set. A quantitative version of the p -version of the above characterization of Pełczyński's property (V^*) is given in Section 4. Moreover, we prove quantitative versions of some results about property p - (V^*) and property p - (V) . It is proved in [31] that the quantities $\omega(\cdot)$, $wk(\cdot)$ and $wck(\cdot)$ are equal in $L_1(\mu)$ for a general positive measure μ . In Section 4, we prove that the quantities $wk(\cdot)$ and $\iota_1(\cdot)$ are equal in $L_1(\mu)$ (μ a finite positive measure) and l_1 . In particular, both $L_1(\mu)$ (μ a finite positive measure) and l_1 have quantitative Pełczyński's property (V^*) with constant 1.

Our notation and terminology are standard as may be found in [1] and [39]. Throughout the paper, all Banach spaces can be considered either real or complex unless stated otherwise. By an operator, we always mean a bounded linear operator. For a Banach space X , $J_X : X \rightarrow X^{**}$ denotes the canonical embedding. p^* will always denote the conjugate number of p for $1 \leq p < \infty$; if $p = 1$, l_{p^*} plays the role of c_0 . Let X be a Banach space, $1 \leq p < \infty$ and denote by $l_p^w(X)$ the space of all weakly p -summable sequences in X , endowed with the norm

$$\|(x_n)_n\|_p^w = \sup \left\{ \left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}, \quad (x_n)_n \in l_p^w(X).$$

It is a well-known result of A. Grothendieck ([27], [21, Proposition 2.2]) that the canonical correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{L}(l_{p^*}, X)$ onto $l_p^w(X)$. We denote by $c_0^w(X)$ the space of weakly null sequences in X . Recall that an operator $T : X \rightarrow Y$ is *completely continuous* if T takes weakly null sequences to norm null sequences. In [10], J. M. F. Castillo introduced the ideal C_p of p -converging operators, which is intermediate between the ideal of unconditionally converging operators and the ideal of completely continuous operators. He defined the ideal of p -converging operators as those operators which transform weakly p -summable sequences into norm null sequences.

2 | PEŁCZYŃSKI'S PROPERTY (V) OF ORDER p AND PEŁCZYŃSKI'S PROPERTY (V^*) OF ORDER p

Definition 2.1. Let $1 \leq p \leq \infty$. We say that a Banach space X has Pełczyński's property (V) of order p (property p - (V) in short) if for every Banach space Y , every p -converging operator $T : X \rightarrow Y$ is weakly compact.

Obviously, for every $1 \leq p < q \leq \infty$, a Banach space X has property q - (V) whenever X has property p - (V) . But we do not know whether there are, for every $1 \leq p < q \leq \infty$, Banach spaces with property q - (V) but not property p - (V) .

Definition 2.2 ([15]). Let X be a Banach space and let $1 \leq p \leq \infty$. We say that a bounded subset K of X^* is a p - (V) set if

$$\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0,$$

for every $(x_n)_n \in l_p^w(X)$ ($(x_n)_n \in c_0^w(X)$ for $p = \infty$).

Before giving a useful characterization of p - (V) sets, we recall the notion of weakly p -convergent sequences introduced in [13]. Let $1 \leq p \leq \infty$. A sequence $(x_n)_n$ in a Banach space X is said to be *weakly p -convergent* to $x \in X$ if the sequence $(x_n - x)_n$ is weakly p -summable in X . Weakly ∞ -convergent sequences are simply the weakly convergent sequences. The concept of weakly p -Cauchy sequences is introduced in [15]. We say that a sequence $(x_n)_n$ in a Banach space X is *weakly p -Cauchy* if for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(x_{k_n} - x_{j_n})_n$ is weakly p -summable

in X . Obviously, every weakly p -convergent sequence is weakly p -Cauchy, and the weakly ∞ -Cauchy sequences are precisely the weakly Cauchy sequences. J. M. F. Castillo and F. Sánchez said that a Banach space $X \in \mathcal{W}_p$ ($1 \leq p \leq \infty$) if any bounded sequence in X admits a weakly p -convergent subsequence (see [13]). The following characterization of p -(V) sets, which will be used in Section 3, appears in [15].

Theorem 2.3 ([15]). *Let $1 < p < \infty$ and let X be a Banach space. The following statements are equivalent about a bounded subset K of X^* :*

- (1) K is a p -(V) set;
- (2) For all spaces $Y \in \mathcal{W}_p$ and for every operator T from Y into X , the subset $T^*(K)$ is relatively norm compact;
- (3) For every operator T from l_{p^*} into X , the subset $T^*(K)$ is relatively norm compact.

In case of $p = 1$, R. Cilia and G. Emmanuele proved that a bounded subset K of X^* is a 1-(V) set if and only if for every operator T from c_0 into X , the subset $T^*(K)$ is relatively norm compact (see [17]). However, the equivalence between (1) and (3) of Theorem 2.3 is false for $p = \infty$ ($p^* = 1$). For instance, by Schur property, every bounded subset of l_∞ is a ∞ -(V) set, but B_{l_∞} is not relatively norm compact.

To show that Pełczyński's property (V) of order p is automatically quantitative, we introduce a quantity $\xi_p(\cdot)$.

Let us fix some notations. If A and B are nonempty subsets of a Banach space X , we set

$$d(A, B) = \inf \{ \|a - b\| : a \in A, b \in B \}, \quad \widehat{d}(A, B) = \sup \{ d(a, B) : a \in A \}.$$

Thus, $d(A, B)$ is the ordinary distance between A and B , and $\widehat{d}(A, B)$ is the non-symmetrized Hausdorff distance from A to B .

Let X be a Banach space and let A be a bounded subset of X^* . For $1 \leq p \leq \infty$, we set

$$\xi_p(A) = \inf \left\{ \widehat{d}(A, K) : K \subset X^* \text{ is a } p\text{-}(V) \text{ set} \right\}.$$

It is clear that $\xi_p(A) = 0$ if and only if A is a p -(V) set.

We also need the de Blasi measure of weak non-compactness introduced by de Blasi in [18].

Let A be a bounded subset of a Banach space X . The de Blasi measure of weak non-compactness of A is defined by

$$\omega(A) = \inf \left\{ \widehat{d}(A, K) : \emptyset \neq K \subset X \text{ is weakly compact} \right\}.$$

It is proved in [18] that $\omega(A) = 0$ if and only if A is relatively weakly compact. For an operator $T : X \rightarrow Y$, we denote $\xi_p(T^*B_{Y^*})$, $\omega(TB_X)$ by $\xi_p(T^*)$, $\omega(T)$ respectively. Then $\xi_p(T^*) = 0$ if and only if T is p -converging, and $\omega(T) = 0$ if and only if T is weakly compact.

Theorem 2.4. *Let $1 \leq p \leq \infty$ and let X be a Banach space. The following are equivalent:*

- (1) X has property p -(V);
- (2) Every p -(V) subset of X^* is relatively weakly compact;
- (3) $\omega(T^*) \leq \xi_p(T^*)$ for every operator T from X into any Banach space Y ;
- (4) $\omega(A) \leq \xi_p(A)$ for every bounded subset A of X^* .

Proof. (1) \Rightarrow (2). Suppose that K is a p -(V) subset of X^* . Take any sequence $(x_n^*)_n$ in K . Define an operator $T : X \rightarrow l_\infty$ by

$$Tx = \left(\langle x_n^*, x \rangle \right)_n, \quad x \in X.$$

Then, for every $(x_n)_n \in l_p^w(X)$, we have

$$\|Tx_n\| = \sup_k |\langle x_k^*, x_n \rangle| \leq \sup_{x^* \in K} |\langle x^*, x_n \rangle| \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus T is p -converging. By (1), the operator T is weakly compact and hence T^* is also weakly compact. This implies that the set $T^*B_{l_\infty^*}$ is relatively weakly compact. It is easy to see that $T^*e_n = x_n^*$ for each $n \in \mathbb{N}$, where $(e_n)_n$ is the unit vector basis of l_1 . So the sequence $(x_n^*)_n$ is relatively weakly compact.

(2) \Rightarrow (3) and (3) \Rightarrow (1) are obvious. The equivalence (2) \Leftrightarrow (4) is straightforward. \square

The following Corollary 2.5 is a generalization of [40, Corollary 1].

Corollary 2.5. *Let $1 \leq p \leq \infty$. If a Banach space X has property p -(V), then every quotient of X has property p -(V).*

Recall that the James p -space J_p ($1 < p < \infty$) is the (real) Banach space of all sequences $(a_n)_n$ of real numbers such that $\lim_{n \rightarrow \infty} a_n = 0$ and

$$\|(a_n)_n\|_{c_{pv}} = \frac{1}{2^{\frac{1}{p}}} \sup \left\{ \left(\sum_{j=1}^m |a_{i_{j-1}} - a_{i_j}|^p + |a_{i_m} - a_{i_0}|^p \right)^{\frac{1}{p}} : 1 \leq i_0 < i_1 < \dots < i_m, m \in \mathbf{N} \right\} < \infty.$$

Another useful equivalent norm on J_p is given by the formula

$$\|(a_n)_n\|_{pv} = \sup \left\{ \left(\sum_{j=1}^m |a_{i_{j-1}} - a_{i_j}|^p \right)^{\frac{1}{p}} : 1 \leq i_0 < i_1 < \dots < i_m, m \in \mathbf{N} \right\}.$$

In fact,

$$\frac{1}{2^{\frac{1}{p}}} \|\cdot\|_{pv} \leq \|\cdot\|_{c_{pv}} \leq \|\cdot\|_{pv}. \tag{2.1}$$

The sequence $(e_n)_n$ of standard unit vectors forms a monotone shrinking basis for J_p in both norms $\|\cdot\|_{pv}$ and $\|\cdot\|_{c_{pv}}$. It is known that J_p is non-reflexive and is codimension of 1 in J_p^{**} , but every infinite-dimensional closed subspace of J_p contains a subspace isomorphic to l_p . We refer to [45] for more information on the James p -space.

The proof of the following lemma is identical to [1, Proposition 3.4.3].

Lemma 2.6. *Let $(x_k)_k$ be a normalized block basic sequence with respect to $(e_n)_n$ in $(J_p, \|\cdot\|_{pv})$. Then, for any sequence $(\lambda_k)_{k=1}^n$ of real numbers and any $n \in \mathbf{N}$ the following estimate holds:*

$$\left\| \sum_{k=1}^n \lambda_k x_k \right\|_{pv} \leq (1 + 2^p)^{\frac{1}{p}} \left(\sum_{k=1}^n |\lambda_k|^p \right)^{\frac{1}{p}}.$$

Theorem 2.7. *The James p -space J_p ($1 < p < \infty$) has property p^* -(V).*

Proof. *Let K be a p^* -(V) subset of $B_{J_p^*}$. Take any sequence $(x_n^*)_n$ from K . Since J_p is separable, we may assume that $(x_n^*)_n$ is weak*-convergent to some $x_0^* \in J_p^*$.*

Claim. $(x_n^*)_n$ converges to x_0^* weakly.

*Note that $J_p^{**} = J_p \oplus \text{span}\{x_0^{**}\}$, where $x_0^{**} \in J_p^{**}$ is defined by $\langle x_0^{**}, e_n^* \rangle = 1$ for all $n \in \mathbf{N}$, where $(e_n^*)_n$ is the sequence of the functionals biorthogonal to the unit vector basis $(e_n)_n$ of J_p . Thus it suffices to prove that*

$$\langle x_0^{**}, x_n^* \rangle \rightarrow \langle x_0^{**}, x_0^* \rangle \quad (n \rightarrow \infty).$$

*Suppose that it is false. By passing to subsequences, we may assume that $|\langle x_0^{**}, x_n^* - x_0^* \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $n \in \mathbf{N}$. Since $(e_n)_n$ is shrinking, $(e_n^*)_n$ forms a basis for J_p^* . Thus*

$$\left| \sum_{k=1}^{\infty} \langle x_n^* - x_0^*, e_k \rangle \right| = \left| \left\langle x_0^{**}, \sum_{k=1}^{\infty} \langle x_n^* - x_0^*, e_k \rangle e_k^* \right\rangle \right| > \epsilon_0, \quad n = 1, 2, \dots \tag{2.2}$$

Note that

$$\lim_{n \rightarrow \infty} \langle x_n^* - x_0^*, e_k \rangle = 0, \quad k = 1, 2, \dots \tag{2.3}$$

By inductions on n in (2.2) and on k in (2.3), we obtain $1 = n_1 < n_2 < n_3 < \dots$ and $0 = k_0 < k_1 < k_2 < k_3 < \dots$ such that

$$\left| \sum_{k=k_{j-1}+1}^{k_j} \langle x_{n_j}^* - x_0^*, e_k \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 1, 2, \dots \quad (2.4)$$

We set $x_j = \sum_{k=k_{j-1}+1}^{k_j} e_k$ ($j = 1, 2, \dots$). Then $(x_j)_j$ is a normalized block basic sequence with respect to $(e_n)_n$ in $(J_p, \|\cdot\|_{c_{pv}})$. It follows from (2.1) and Lemma 2.6 that for any sequence of real scalars $(\lambda_j)_{j=1}^n$ the following estimate holds:

$$\left\| \sum_{j=1}^n \lambda_j x_j \right\|_{c_{pv}} \leq (2 + 2^{p+1})^{\frac{1}{p}} \left(\sum_{j=1}^n |\lambda_j|^p \right)^{\frac{1}{p}}$$

It follows from the fact $\mathcal{L}(l_p, J_p) = l_p^w(J_p)$ that $(x_j)_j$ is weakly p^* -summable. Since K is a p^* -(V) set, we get

$$\left| \langle x_{n_j}^*, x_j \rangle \right| \leq \sup_{x^* \in K} |\langle x^*, x_j \rangle| \rightarrow 0 \quad (j \rightarrow \infty).$$

Obviously, $(\langle x_0^*, x_j \rangle)_j$ converges to 0. Therefore, we have

$$\left| \langle x_{n_j}^* - x_0^*, x_j \rangle \right| \rightarrow 0 \quad (j \rightarrow \infty),$$

which contradicts (2.4). This contradiction shows that $(x_n^*)_n$ converges to x_0^* weakly. Thus K is relatively weakly compact. By Theorem 2.4, J_p has property p^* -(V). \square

As in [12], we consider the space X_p ($1 < p < \infty$) constructed in [25]. We do not describe the space X_p here and refer the reader to [25] for details. In [25], a quotient map $T_p : X_p \rightarrow c_0$ is defined and it is proved that T_p is unconditionally converging. We extend this result as follows:

Lemma 2.8. *The quotient map T_p is q -converging for any $1 \leq q < p^*$.*

Proof. Suppose that T_p is not q -converging for some $1 \leq q < p^*$. It follows from [11, Remark before Proposition 1.6] that there exists an operator S from l_{q^*} (c_0 for $q = 1$) into X_p such that $T_p S$ is non-compact. Thus, we can find a weakly null sequence $(z_n)_n$ in l_{q^*} and $\epsilon_0 > 0$ such that $\|T_p S z_n\| \geq \epsilon_0$ for each $n \in \mathbb{N}$. By passing to subsequences, we may assume that $(z_n)_n$ is equivalent to the unit vector basis of l_{q^*} , that is, there exist $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \alpha_2, \dots, \alpha_n$, one has

$$C_1 \left(\sum_{k=1}^n |\alpha_k|^{q^*} \right)^{\frac{1}{q^*}} \leq \left\| \sum_{k=1}^n \alpha_k z_k \right\| \leq C_2 \left(\sum_{k=1}^n |\alpha_k|^{q^*} \right)^{\frac{1}{q^*}}. \quad (2.5)$$

Let $x_n = S z_n$. By [26, Proposition 2], the sequence $(x_n)_n$ admits a subsequence, which is still denoted by $(x_n)_n$, such that $(x_{2n-1} - x_{2n})_n$ is equivalent to the unit vector basis of l_p . Then, there exist $D_1, D_2 > 0$ such that for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \alpha_2, \dots, \alpha_n$, one has

$$D_1 \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{k=1}^n \alpha_k (x_{2k-1} - x_{2k}) \right\| \leq D_2 \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}}. \quad (2.6)$$

By (2.5) and (2.6), we get, for each n and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$D_1 \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{k=1}^n \alpha_k (x_{2k-1} - x_{2k}) \right\| \leq \|S\| \cdot \left\| \sum_{k=1}^n \alpha_k (z_{2k-1} - z_{2k}) \right\| \leq \|S\| \cdot C_2 \cdot 2^{\frac{1}{q^*}} \left(\sum_{k=1}^n |\alpha_k|^{q^*} \right)^{\frac{1}{q^*}},$$

which is impossible because $1 \leq q < p^*$. This completes the proof. \square

Theorem 2.9. *Property q -(V) is not a three-space property for each $1 \leq q < \infty$.*

Proof. For $1 \leq q < \infty$, choose $1 < p < \infty$ with $q < p^*$. It is shown in [12] that both $X_p/\text{Ker}(T_p)$ and $\text{Ker}(T_p)$ have property $1-(V)$ and hence have property $q-(V)$. But X_p fails property $q-(V)$ since, by Lemma 2.8, T_p is q -converging, but obviously not weakly compact. \square

Definition 2.10. Let X be a Banach space and let $1 \leq p \leq \infty$. We say that a bounded subset K of X is a $p-(V^*)$ set if

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle x_n^*, x \rangle| = 0,$$

for every $(x_n^*)_n \in l_p^w(X^*)$ ($(x_n^*)_n \in c_0^w(X^*)$ for $p = \infty$).

Theorem 2.11. Let K be a bounded subset of a Banach space X and let $1 < p < \infty$. The following statements are equivalent:

- (1) K is a $p-(V^*)$ set;
- (2) For all spaces Y with $Y^* \in W_p$, every operator $T : X \rightarrow Y$ maps K onto a relatively norm compact subset of Y ;
- (3) Every operator $T : X \rightarrow l_p$ maps K onto a relatively norm compact subset of l_p .

Proof. (1) \Rightarrow (2). Let Y and T be as stated in (2). Assume to the contrary that $T(K)$ is not relatively norm compact. Then there exists a sequence $(x_n)_n$ in K such that $(Tx_n)_n$ admits no norm convergent subsequences. Since Y is reflexive, by passing to a subsequence if necessary we may assume that $(Tx_n)_n$ converges weakly to some $y \in Y$ and $\|Tx_n - y\| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, choose y_n^* with $\|y_n^*\| \leq 1$ such that $|\langle y_n^*, Tx_n - y \rangle| > \epsilon_0$. Since $Y^* \in W_p$, by passing to a subsequence again one can assume that the sequence $(y_n^*)_n$ is weakly p -convergent to some $y^* \in Y^*$. By (1), we get

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle T^* y_n^* - T^* y^*, x \rangle| = 0.$$

For each $n \in \mathbf{N}$, we have

$$\begin{aligned} \epsilon_0 < |\langle y_n^*, Tx_n - y \rangle| &\leq |\langle y_n^*, Tx_n \rangle - \langle y^*, Tx_n \rangle| + |\langle y^*, Tx_n \rangle - \langle y^*, y \rangle| + |\langle y^*, y \rangle - \langle y_n^*, y \rangle| \\ &\leq \sup_{x \in K} |\langle T^* y_n^* - T^* y^*, x \rangle| + |\langle y^*, Tx_n - y \rangle| + |\langle y^* - y_n^*, y \rangle| \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

which is a contradiction.

(2) \Rightarrow (3) is immediate because $l_p^* \in W_p$:

(3) \Rightarrow (1). Let $(x_n^*)_n \in l_p^w(X^*)$. Then there exists an operator T from X into l_p such that $Tx = (\langle x_n^*, x \rangle)_n$ for all $x \in X$. It follows from (3) that $T(K)$ is relatively norm compact. By the well-known characterization of relatively norm compact subsets of l_p (see [20, p. 6]), one can derive that $\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle x_n^*, x \rangle| = 0$. This finishes the proof. \square

It should be mentioned that G. Emmanuele proved the equivalence between (1) and (3) of Theorem 2.11 for $p = 1$ (see [23]). Obviously, this is false for $p = \infty$, for example, take $X = c_0$. But, K. T. Andrews proved that a bounded subset K of a Banach space X is a $\infty-(V^*)$ set if and only if every weakly compact operator $T : X \rightarrow c_0$ maps K onto a relatively norm compact subset (see [3]).

Definition 2.12. Let $1 \leq p \leq \infty$. We say that a Banach space X has Pełczyński's property (V^*) of order p (property $p-(V^*)$ in short) if every $p-(V^*)$ subset of X is relatively weakly compact.

It is clear that for every $1 \leq p < q \leq \infty$, a Banach space X has property $q-(V^*)$ whenever X has property $p-(V^*)$. Likewise, we still do not know whether there are, for every $1 \leq p < q \leq \infty$, Banach spaces with property $q-(V^*)$ but not property $p-(V^*)$.

The proof of the following lemma is similar to [2, Proposition 5].

Lemma 2.13. Let $(x_n^*)_n = \left(\sum_{i=k_{n-1}+1}^{k_n} a_i e_i^* \right)_n$ be a semi-normalized block basic sequence with respect to $(e_n^*)_n$ in J_p^* and suppose that $\sum_{i=k_{n-1}+1}^{k_n} a_i = 0$ for each $n \in \mathbf{N}$. Then $(x_n^*)_n$ is equivalent to the unit vector basis of l_p^* .

Theorem 2.14. The James p -space J_p ($1 < p < \infty$) has property $p-(V^*)$.

Proof. Let K be a $p-(V^*)$ subset of B_{J_p} . Take any sequence $(x_n)_n$ from K . Since J_p^* is separable, we may assume that $(x_n)_n$ is weak*-convergent to some $x^{**} \in B_{J_p^{**}}$. We aim to prove that $x^{**} \in J_p$, that is, $\lim_{k \rightarrow \infty} \langle x^{**}, e_k^* \rangle = \xi = 0$.

Suppose that $\xi \neq 0$. Let $\delta = \frac{|\xi|}{2} > 0$. Then there exists $p_1 \in \mathbf{N}$ such that $|\langle x^{**}, e_k^* \rangle| > \delta$ for all $k \geq p_1$. Since $(x_n)_n$ is weak*-convergent to x^{**} , we choose n_1 such that $|\langle e_{p_1}^*, x_{n_1} \rangle| > \delta$. Choose $q_1 > p_1$ such that $|\langle e_k^*, x_{n_1} \rangle| < \frac{\delta}{2}$ for each $k \geq q_1$. In

particular, $\left| \langle e_{q_1}^*, x_{n_1} \rangle \right| < \frac{\delta}{2}$. Choose any $p_2 > q_1$. Then there exists $n_2 > n_1$ such that $\left| \langle e_{p_2}^*, x_{n_2} \rangle \right| > \delta$. Choose $q_2 > p_2$ such that $\left| \langle e_k^*, x_{n_2} \rangle \right| < \frac{\delta}{2}$ for each $k \geq q_2$. In particular, $\left| \langle e_{q_2}^*, x_{n_2} \rangle \right| < \frac{\delta}{2}$. We continue in a similar manner and obtain

$$p_1 < q_1 < p_2 < q_2 < \dots \quad \text{and} \quad n_1 < n_2 < \dots$$

such that

$$\left| \langle e_{p_j}^*, x_{n_j} \rangle \right| > \delta \quad \text{and} \quad \left| \langle e_{q_j}^*, x_{n_j} \rangle \right| < \frac{\delta}{2}, \quad j = 1, 2, \dots$$

Set $z_j^* = e_{p_j}^* - e_{q_j}^*$ ($j = 1, 2, \dots$). Then, for each $j \in \mathbf{N}$, we have

$$\left| \langle z_j^*, x_{n_j} \rangle \right| = \left| \langle e_{p_j}^*, x_{n_j} \rangle - \langle e_{q_j}^*, x_{n_j} \rangle \right| \geq \left| \langle e_{p_j}^*, x_{n_j} \rangle \right| - \left| \langle e_{q_j}^*, x_{n_j} \rangle \right| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

Thus $(z_j^*)_j$ is a semi-normalized block basic sequence of $(e_n^*)_n$. It follows from Lemma 2.13 that $(z_j^*)_j$ is equivalent to the unit vector basis of l_{p^*} . In particular, $(z_j^*)_j$ is weakly p -summable. Since K is a p - (V^*) set, we get

$$\frac{\delta}{2} < \left| \langle z_j^*, x_{n_j} \rangle \right| \leq \sup_{x \in K} \left| \langle z_j^*, x \rangle \right| \rightarrow 0 \quad (j \rightarrow \infty),$$

which is a contradiction. □

The following Corollary 2.15 is an extension of [40, Proposition 5].

Corollary 2.15. *Let $1 \leq p \leq \infty$. If a Banach space X has property p - (V^*) , then every closed subspace of X has property p - (V^*) .*

The following Corollary 2.16 is a generalization of [40, Proposition 4].

Corollary 2.16. *Let $1 \leq p \leq \infty$ and let X be a Banach space. Then*

- (1) *If X has property p - (V) , then X^* has property p - (V^*) ;*
- (2) *If X^* has property p - (V) , then X has property p - (V^*) .*

We remark that the converse of Corollary 2.16 is not true for all $1 \leq p \leq \infty$. J. Bourgain and F. Delbaen (see [8]) constructed a Banach space X_{BD} such that X_{BD} has the Schur property, X_{BD}^* is isomorphic to an L_1 -space. Thus, the space X_{BD} fails property p - (V) for all $1 \leq p \leq \infty$. Since X_{BD}^* is isomorphic to an L_1 -space, X_{BD}^* has property 1- (V^*) and hence property p - (V^*) for all $1 \leq p \leq \infty$.

Definition 2.17. *Let $1 \leq p \leq \infty$. We say that a Banach space X is weakly sequentially complete of order p if every weakly p -Cauchy sequence in X is weakly p -convergent.*

The weakly sequential completeness of order ∞ is precisely the classical weakly sequential completeness. It is easy to verify that for $1 \leq p < q \leq \infty$, a Banach space X is weakly sequentially complete of order p whenever X is weakly sequentially complete of order q .

Theorem 2.18. *Let $1 < p < 2$. If a Banach space X has property p - (V^*) , then X is weakly sequentially complete of order p .*

Proof. *It follows from Pitt's Theorem and [11, Remark before Proposition 1.6] that the identity $I_p : l_p \rightarrow l_p$ is p -converging. By [15, Theorem 2.2], we see that every weakly p -Cauchy sequence in l_p is convergent in norm. Let $(x_n)_n$ be a weakly p -Cauchy sequence in X . Then, for every operator $T : X \rightarrow l_p$, the sequence $(Tx_n)_n$ is weakly p -Cauchy and hence is convergent in norm. By Theorem 2.11, we get that $(x_n)_n$ is a p - (V^*) set. Since X has property p - (V^*) , the sequence $(x_n)_n$ is relatively weakly compact. Thus, $(x_n)_n$ is weakly p -convergent. □*

It is worth mentioning that we do not know whether Theorem 2.18 is true for all $1 < p < \infty$.

Corollary 2.19. *Let $1 < p < 2$. If a Banach space X has property p - (V) , then X^* is weakly sequentially complete of order p .*

3 | PEŁCZYŃSKI'S PROPERTY (V) OF ORDER P AND PEŁCZYŃSKI'S PROPERTY (V*) OF ORDER P IN VECTOR-VALUED SEQUENCE SPACES

Let $(X_n)_n$ be a sequence of Banach spaces and $1 \leq p < \infty$. We denote by $(\sum_{n=1}^{\infty} \oplus X_n)_p$ the space of all vector-valued sequences $x = (x_n)_n$ with $x_n \in X_n (n \in \mathbf{N})$, for which

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} < \infty.$$

Similarly, $(\sum_{n=1}^{\infty} \oplus X_n)_0$ denotes the space of all vector-valued sequences $x = (x_n)_n$ with $x_n \in X_n (n \in \mathbf{N})$, for which $\lim_{n \rightarrow \infty} \|x_n\| = 0$, endowed with the supreme norm. The direct sum in the sense of l_{∞} of $(X_n)_n$, denoted by $(\sum_{n=1}^{\infty} \oplus X_n)_{\infty}$, is defined in an analogous way. For every $n \in \mathbf{N}$, I_n will denote the canonical injection from X_n into $(\sum_{n=1}^{\infty} \oplus X_n)_p$ and π_n will denote the canonical projection from $(\sum_{n=1}^{\infty} \oplus X_n)_p$ onto X_n . We denote the canonical injection J_n from X_n^* into $(\sum_{n=1}^{\infty} \oplus X_n^*)_{p^*}$ and the canonical projection from $(\sum_{n=1}^{\infty} \oplus X_n^*)_{p^*}$ onto X_n^* by P_n . Clearly, $I_n^* = P_n$ and $\pi_n^* = J_n$.

Theorem 3.1. *Let $(X_n)_n$ be a sequence of Banach spaces and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_p (1 < p < \infty)$ or $X = (\sum_{n=1}^{\infty} \oplus X_n)_0$. The following are equivalent for a bounded subset A of X^* :*

- (1) A is a p^* -(V) set;
- (2) $P_n(A)$ is a p^* -(V) set for each $n \in \mathbf{N}$ and

$$\limsup_{n \rightarrow \infty} \left\{ \sum_{k=n}^{\infty} \|P_k x^*\|^{p^*} : x^* \in A \right\} = 0.$$

Proof. (1) \Rightarrow (2). *It is obvious that $P_n(A)$ is a p^* -(V) set for each $n \in \mathbf{N}$. Let us assume that*

$$\limsup_{n \rightarrow \infty} \left\{ \sum_{k=n}^{\infty} \|P_k x^*\|^{p^*} : x^* \in A \right\} \neq 0.$$

By induction, we can find $\epsilon_0 > 0$, two sequences of positive integers $(p_n)_n, (q_n)_n$ with $p_n < q_n < p_{n+1} (n \in \mathbf{N})$ and a sequence $(x_k^{(n)})_k$ in A such that $\sum_{k=p_n}^{q_n} \|P_k x_k^{(n)}\|^{p^} > \epsilon_0$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, there exists a sequence $(x_k^{(n)})_{k=p_n}^{q_n} \in (\sum_{k=p_n}^{q_n} \oplus X_k)_p$ such that $\sum_{k=p_n}^{q_n} \|x_k^{(n)}\|^p = 1$ and*

$$\sum_{k=p_n}^{q_n} \langle P_k x_k^{(n)}, x_k^{(n)} \rangle = \left(\sum_{k=p_n}^{q_n} \|P_k x_k^{(n)}\|^{p^*} \right)^{\frac{1}{p^*}} > \epsilon_0^{\frac{1}{p^*}}. \quad (3.1)$$

For every $n \in \mathbf{N}$, we set $f_n \in X = (\sum_{k=1}^{\infty} \oplus X_k)_p$ by

$$\pi_k(f_n) = \begin{cases} x_k^{(n)}, & p_n \leq k \leq q_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then the sequence $(f_n)_n$ is weakly p^ -summable. Indeed, for every $x^* \in X^*$, we have*

$$|\langle x^*, f_n \rangle| = \left| \sum_{k=p_n}^{q_n} \langle P_k x^*, x_k^{(n)} \rangle \right| \leq \left(\sum_{k=p_n}^{q_n} \|x_k^{(n)}\|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{k=p_n}^{q_n} \|P_k x^*\|^{p^*} \right)^{\frac{1}{p^*}} = \left(\sum_{k=p_n}^{q_n} \|P_k x^*\|^{p^*} \right)^{\frac{1}{p^*}},$$

which implies

$$\sum_{n=1}^{\infty} |\langle x^*, f_n \rangle|^{p^*} \leq \sum_{n=1}^{\infty} \sum_{k=p_n}^{q_n} \|P_k x^*\|^{p^*} \leq \sum_{k=1}^{\infty} \|P_k x^*\|^{p^*} < \infty.$$

By (1), we get

$$\sum_{k=p_n}^{q_n} \langle P_k x_n^*, x_k^{(n)} \rangle = |\langle x_n^*, f_n \rangle| \leq \sup_{x^* \in A} |\langle x^*, f_n \rangle| \rightarrow 0 \quad (n \rightarrow \infty),$$

which contradicts (3.1).

(2) \Rightarrow (1). Let T be an operator from $l_p(c_0$ for $p^* = 1)$ into X . Then, by (2), we have

$$\sup_{x^* \in A} \left\| \sum_{k=1}^n T^* \circ J_k \circ P_k x^* - T^* x^* \right\| \leq \|T\| \cdot \sup_{x^* \in A} \left(\sum_{k=n+1}^{\infty} \|P_k x^*\|^{p^*} \right)^{\frac{1}{p^*}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$T^* A \subset \sum_{k=1}^{n_0} T^* \circ J_k \circ P_k A + \epsilon B_{l_{p^*}}.$$

Since $P_k(A)$ is a p^* -(V) set for each $k = 1, 2, \dots, n_0$, we get, by Theorem 2.3, that the subset $T^* \circ \pi_k^* \circ P_k A$ is relatively norm compact for each $k = 1, 2, \dots, n_0$ and so is $\sum_{k=1}^{n_0} T^* \circ J_k \circ P_k A$. Therefore, the subset $T^* A$ is relatively norm compact. Again by Theorem 2.3, we see that A is a p^* -(V) set. \square

Theorem 3.2. Let $(X_n)_n$ be a sequence of Banach spaces, let $1 < p < \infty$, $1 \leq q < p^*$, and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$. Then a bounded subset A of X^* is a q -(V) set if and only if each $P_n(A)$ is.

Proof. We need only prove the sufficient part. Assume that A is not a q -(V) set. Then there exist $\epsilon_0 > 0$, a sequence $(x_n)_n \in l_q^w(X)$ and a sequence $(x_n^*)_n$ in A such that

$$|\langle x_n^*, x_n \rangle| = \left| \sum_{k=1}^{\infty} \langle P_k x_n^*, \pi_k x_n \rangle \right| > \epsilon_0, \quad n = 1, 2, \dots \tag{3.2}$$

By the assumption, we get

$$\lim_{n \rightarrow \infty} \langle P_k x_n^*, \pi_k x_n \rangle = 0, \quad k = 1, 2, \dots \tag{3.3}$$

By induction on n in (3.2) and k in (3.3), we get

$$1 = n_1 < n_2 < \dots, \quad 0 = k_0 < k_1 < k_2 < \dots,$$

such that

$$\left| \sum_{k=k_j+1}^{\infty} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| < \frac{\epsilon_0}{4}, \quad j = 1, 2, \dots, \tag{3.4}$$

and

$$\left| \sum_{k=1}^{k_j} \langle P_k x_{n_{j+1}}^*, \pi_k x_{n_{j+1}} \rangle \right| < \frac{\epsilon_0}{4}, \quad j = 1, 2, \dots \tag{3.5}$$

By (3.2), (3.4) and (3.5), we get

$$\left| \sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 2, 3, \dots$$

By (3.2) and (3.4), we get

$$\left| \sum_{k=1}^{k_1} \langle P_k x_{n_1}^*, \pi_k x_{n_1} \rangle \right| > \frac{3}{4} \epsilon_0 > \frac{\epsilon_0}{2}.$$

Thus, we have

$$\left| \sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 1, 2, \dots$$

For each $j = 1, 2, \dots$, we set $y_j = x_{n_j}$ and $y_j^* \in X^*$ by

$$P_k y_j^* = \begin{cases} P_k x_{n_j}^*, & k_{j-1} + 1 \leq k \leq k_j, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $(y_j)_j \in l_q^w(X)$ and

$$|\langle y_j^*, y_j \rangle| = \left| \sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 1, 2, \dots$$

Since the sequence $(y_j^*)_j$ has pairwise disjoint supports, we see that $(y_j^*)_j$ is equivalent to the unit vector basis $(e_j)_j$ of l_{p^*} . Let R be an isomorphic embedding from l_{p^*} into X^* with $Re_j = y_j^* (j = 1, 2, \dots)$. Let T be an operator from l_{q^*} into X . By Pitt's theorem, the operator T^*R is compact and hence the sequence $(T^*y_j^*)_j = (T^*Re_j)_j$ is relatively norm compact. It follows from Theorem 2.3 that the sequence $(y_j^*)_j$ is a q -(V) set. Since $(y_j)_j$ is weakly q -summable, we have

$$|\langle y_n^*, y_n \rangle| \leq \sup_j |\langle y_j^*, y_n \rangle| \rightarrow 0 \quad (n \rightarrow \infty),$$

this contradiction concludes the proof. □

The following two lemmas are wellknown (see [6], for example).

Lemma 3.3. Let $(X_n)_n$ be a sequence of Banach spaces. The following are equivalent about a bounded subset A of $(\sum_{n=1}^\infty \oplus X_n)_1$:

- (1) A is relatively weakly compact;
- (2) $\pi_n(A)$ is relatively weakly compact for each $n \in \mathbb{N}$ and

$$\limsup_{n \rightarrow \infty} \left\{ \sum_{k=n}^\infty \|\pi_k x\| : x \in A \right\} = 0.$$

Lemma 3.4. Let $(X_n)_n$ be a sequence of Banach spaces and let $X = (\sum_{n=1}^\infty \oplus X_n)_p (1 < p < \infty)$ or $X = (\sum_{n=1}^\infty \oplus X_n)_0$. Then a bounded subset A of X is relatively weakly compact if and only if every $\pi_n(A)$ is.

Theorem 3.5. Let $(X_n)_n$ be a sequence of Banach spaces, let $1 \leq q < \infty$ and $1 < p < \infty$. Then $(\sum_{n=1}^\infty \oplus X_n)_p$ has property q -(V) if and only if each X_n is.

Proof. The necessary part follows from Corollary 2.5.

Conversely, let A be a q -(V) subset of $(\sum_{n=1}^\infty \oplus X_n)_p$. Then each $P_n A$ is also a q -(V) set. By hypothesis, each $P_n A$ is relatively weakly compact. It follows from Lemma 3.4 that A is relatively weakly compact. This concludes the proof. □

Theorem 3.6. Let $(X_n)_n$ be a sequence of Banach spaces. Then $(\sum_{n=1}^\infty \oplus X_n)_0$ has property 1-(V) if and only if each X_n is.

Proof. The necessary part follows from Corollary 2.5.

Conversely, assume that A is a 1-(V) subset of $(\sum_{n=1}^\infty \oplus X_n)_1$. By Theorem 3.1, each $P_n(A)$ is a 1-(V) set and

$$\limsup_{n \rightarrow \infty} \left\{ \sum_{k=n}^\infty \|P_k x^*\| : x^* \in A \right\} = 0.$$

By the assumption, each $P_n A$ is relatively weakly compact. It follows from Lemma 3.3 that A is relatively weakly compact. We are done. □

Combining Theorem 2.11 with the same argument as Theorem 3.1, we obtain the similar result for the p - (V^*) sets.

Theorem 3.7. *Let $(X_n)_n$ be a sequence of Banach spaces. Let A be a bounded subset of $X = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_p$ ($1 \leq p < \infty$). The following assertions are equivalent:*

- (1) A is a p - (V^*) set;
- (2) $\pi_n(A)$ is a p - (V^*) set for each $n \in \mathbf{N}$ and

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k x\|^p : x \in A \right\} = 0.$$

Theorem 3.8. *Let $(X_n)_n$ be a sequence of Banach spaces and $1 \leq q < p < \infty$. Let A be a bounded subset of $X = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_p$ or $X = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_0$. Then A is a q - (V^*) set if and only if each $\pi_n(A)$ is.*

The proof is similar to Theorem 3.2, only interchanging the role of X and X^* and replacing Theorem 2.3 by Theorem 2.11.

Theorem 3.9. *Let $(X_n)_n$ be a sequence of Banach spaces, let $1 \leq q < \infty$, $1 < p < \infty$ and let $X = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_p$ or $X = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_0$. Then X has property q - (V^*) if and only if each X_n is.*

Proof. *The necessary part follows from Corollary 2.15.*

Conversely, let A be a q - (V^) subset of X . Clearly, each $\pi_n(A)$ is a q - (V^*) subset of X_n . By the assumption, each $\pi_n(A)$ is relatively weakly compact. It follows from Lemma 3.4 that A is relatively weakly compact. Therefore, X has property q - (V^*) . \square*

4 | QUANTITATIVE PEŁCZYŃSKI'S PROPERTY (V) OF ORDER P AND PEŁCZYŃSKI'S PROPERTY (V^*) OF ORDER P

We will need several measures of weak non-compactness. Let A be a bounded subset of a Banach space X . Other commonly used quantities measuring weak non-compactness are:

$$wk_X(A) = \hat{d}(\overline{A}^{w^*}, X), \text{ where } \overline{A}^{w^*} \text{ denotes the weak}^* \text{ closure of } A \text{ in } X^{**}.$$

$wck_X(A) = \sup\{d(\text{clust}_{X^{**}}((x_n)_n), X) : (x_n)_n \text{ is a sequence in } A\}$, where $\text{clust}_{X^{**}}((x_n)_n)$ is the set of all weak* cluster points in X^{**} of $(x_n)_n$.

$$\gamma_X(A) = \sup \left\{ \left| \lim_n \lim_m \langle x_m^*, x_n \rangle - \lim_m \lim_n \langle x_m^*, x_n \rangle \right| : (x_n)_n \text{ is a sequence in } A, (x_m^*)_m \text{ is a sequence in } B_{X^*} \right. \\ \left. \text{and all the involved limits exist} \right\}.$$

It follows from [4, Theorem 2.3] that for any bounded subset A of a Banach space X we have

$$wck_X(A) \leq wk_X(A) \leq \gamma_X(A) \leq 2wck_X(A), \quad wk_X(A) \leq \omega(A). \quad (4.1)$$

For an operator $T : X \rightarrow Y$, $\omega(T)$, $wk_Y(T)$, $wck_Y(T)$, $\gamma_Y(T)$ will denote $\omega(TB_X)$, $wk_Y(TB_X)$, $wck_Y(TB_X)$ and $\gamma_Y(TB_X)$, respectively. C. Angosto and B. Cascales [4] proved the following inequality:

$$\gamma_Y(T) \leq \gamma_{X^*}(T^*) \leq 2\gamma_Y(T).$$

So, putting these inequalities together, we get

$$\frac{1}{2}wk_Y(T) \leq wk_{X^*}(T^*) \leq 4wk_Y(T). \quad (4.2)$$

For an operator $T : X \rightarrow Y$ and $1 \leq p \leq \infty$. We set

$$uc_p(T) = \sup \left\{ \limsup_n \|Tx_n\| : (x_n)_n \in l_p^w(X), \|(x_n)_n\|_p^w \leq 1 \right\}.$$

Then $uc_p(T) = 0$ if and only if T is p -converging.

It should be mentioned that a quantity is defined in [36] to measure how far an operator is unconditionally converging as follows:

$$uc(T) = \sup \left\{ ca \left(\left(\sum_{i=1}^n Tx_i \right)_n \right) : (x_n)_n \in l_1^w(X), \|(x_n)_n\|_1^w \leq 1 \right\},$$

where $ca \left(\left(\sum_{i=1}^n Tx_i \right)_n \right) = \inf_n \sup \left\{ \left\| \sum_{i=1}^k Tx_i - \sum_{i=1}^l Tx_i \right\| : k, l \geq n \right\}$. This quantity is a measure of non-Cauchyness of the sequence $\left(\sum_{i=1}^n Tx_i \right)_n$. More precisely, $ca \left(\left(\sum_{i=1}^n Tx_i \right)_n \right) = 0$ if and only if $\left(\sum_{i=1}^n Tx_i \right)_n$ is norm Cauchy.

It is noted in [36] that $uc(T) = 0$ if and only if T is unconditionally converging. A routine argument shows that $uc_1(T) = uc(T)$.

We begin this section with a simple lemma in [15], which will be used frequently.

Lemma 4.1 ([15]). *Let X be a closed subspace of a Banach space Y and let A be a bounded subset of X . Then*

$$wk_Y(A) \leq wk_X(A) \leq 2wk_Y(A). \quad (4.3)$$

It is worth mentioning that the constant 2 in the right hand side inequality of (4.3) is optimal. Indeed, let $X = c_0$, $Y = l_\infty$ and let A be the summing basis of c_0 . It is easy to check that $wk_X(A) = 1$ and $wk_Y(A) = \frac{1}{2}$.

Definition 4.2. *Let $1 \leq p \leq \infty$. We say that a Banach space X has quantitative Pelczyński's property (V^*) of order p (property $p-(V^*)_q$ in short) with a constant $C > 0$ if for every Banach space Y and every operator $T : Y \rightarrow X$, one has*

$$wk_X(T) \leq C \cdot uc_p(T^*).$$

We say that a Banach space X has property $p-(V^*)_q$ if it has property $p-(V^*)_q$ with some constant C .

Let $1 \leq p \leq \infty$ and let X be a Banach space. For a bounded subset A of X , we set

$$i_p(A) = \sup \left\{ \limsup_n \sup_{x \in A} |\langle x_n^*, x \rangle| : (x_n^*)_n \in l_p^w(X^*) (c_0^w(X^*) \text{ for } p = \infty), \|(x_n^*)_n\|_p^w \leq 1 \right\}.$$

Obviously, $i_p(A) = 0$ if and only if A is a $p-(V^*)_q$ set.

Theorem 4.3. *Let X be a Banach space and let $1 \leq p < \infty$. The following statements are equivalent:*

- (1) X has property $p-(V^*)_q$;
- (2) there exists a constant $C > 0$ such that for each bounded subset A of X , one has

$$wk_X(A) \leq C \cdot i_p(A).$$

Proof. (1) \Rightarrow (2). Suppose that X has property $p-(V^*)_q$ with a constant $C > 0$. Let A be a bounded subset of X . We first claim that

$$wck_X(A) \leq C \cdot i_p(A).$$

Indeed, we may assume that $wck_X(A) > 0$ and fix an arbitrary $\epsilon \in (0, wck_X(A))$. Then there exists a sequence $(x_n)_n$ in A such that $\epsilon < wck_X((x_n)_n)$. Define an operator

$$T : l_1 \rightarrow X, \quad (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n x_n, \quad (\alpha_n)_n \in l_1.$$

By (1) and (4.1), we get

$$wck_X((x_n)_n) \leq wck_X(T) \leq wk_X(T) \leq C \cdot uc_p(T^*).$$

By the definitions of T and $i_p(A)$, we get

$$uc_p(T^*) \leq i_p(A).$$

This yields $\epsilon < C \cdot i_p(A)$. By the arbitrariness of $\epsilon \in (0, wk_X(A))$, we prove the claim. Again by (4.1), we obtain

$$wk_X(A) \leq 2C \cdot i_p(A).$$

(2) \Rightarrow (1) is trivial by taking $A = TB_Y$ with the same constant C . □

It is shown in [44] that property (V^*) is separably determined. That is, a Banach space X has property (V^*) if and only if each of its separable closed subspaces has property (V^*) . We provide a quantitative p -version of this result as follows.

Theorem 4.4. *Let $1 \leq p \leq \infty$. Then a Banach space X has property p - $(V^*)_q$ if and only if there exists a constant $C > 0$ such that every separable closed subspace of X has property p - $(V^*)_q$ with C .*

Proof. *Suppose that X has property p - $(V^*)_q$ with a constant $C > 0$. Let M be a closed subspace of X . We'll show that M has property p - $(V^*)_q$ with $2C$. Fix a Banach space Y and an operator $S : Y \rightarrow M$. Then we have*

$$wk_X(iS) \leq C \cdot uc_p(S^*i^*),$$

where $i : M \rightarrow X$ is the inclusion map. By (4.3), we get

$$wk_M(S) \leq 2wk_X(iS) \leq 2C \cdot uc_p(S^*i^*) \leq 2C \cdot uc_p(S^*),$$

we are done.

Conversely, let $C > 0$ be such that every separable closed subspace of X has property p - $(V^*)_q$ with C . We claim that

$$\gamma_X(T) \leq 2C \cdot uc_p(T^*), \text{ for every Banach space } Y \text{ and every operator } T : Y \rightarrow X.$$

Fix a space Y and an operator $T : Y \rightarrow X$. Let $A = TB_Y$. We may assume that $\gamma_X(A) > 0$. Fix any $\epsilon \in (0, \gamma_X(A))$. Then there exists a sequence $(x_n)_n$ in A such that $\epsilon < \gamma_X((x_n)_n)$. By [29, Proposition 3.4], there exist a separable closed subspace Z of X that contains $(x_n)_n$ and an isometric embedding $J : Z^* \rightarrow X^*$ such that $Jz^*|_Z = z^*$ for every $z^* \in Z^*$. Define an operator

$$P : X^* \rightarrow J(Z^*), \quad x^* \mapsto J(x^*|_Z), \quad x^* \in X^*.$$

Then P is a linear projection from X^* onto $J(Z^*)$ with $\|P\| = 1$. We define an operator

$$S : l_1 \rightarrow Z, \quad (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n x_n, \quad (\alpha_n)_n \in l_1.$$

By hypothesis, we get

$$wk_Z(S) \leq C \cdot uc_p(S^*).$$

By (4.1), we have

$$wk_Z(S) \geq wk_Z((x_n)_n) \geq \frac{1}{2} \gamma_Z((x_n)_n).$$

By the definition of S and the properties of J , we obtain

$$\begin{aligned} \gamma_Z((x_n)_n) &\leq 2C \cdot \sup \left\{ \limsup_n \sup_k |\langle z_n^*, x_k \rangle| : (z_n^*)_n \in l_p^w(Z^*), \| (z_n^*)_n \|_p^w \leq 1 \right\} \\ &\leq 2C \cdot \sup \left\{ \limsup_n \sup_k |\langle x_n^*, x_k \rangle| : (x_n^*)_n \in l_p^w(X^*), \| (x_n^*)_n \|_p^w \leq 1 \right\} \\ &\leq 2C \cdot uc_p(T^*). \end{aligned}$$

By the definition of P , we get

$$\gamma_X((x_n)_n) = \gamma_Z((x_n)_n).$$

Thus, one has $\epsilon < 2C \cdot uc_p(T^*)$, which proves the claim by the arbitrariness of ϵ .

By (4.1) again, we have

$$wk_X(T) \leq 2C \cdot uc_p(T^*).$$

This implies that X has property $p\text{-}(V^*)_q$ with $2C$. □

Theorem 4.5. *Let $X = L_1(\mu, \mathbf{R})$, where (Ω, Σ, μ) is a finite measure space. Then*

$$wk_X(A) = \iota_1(A)$$

for each bounded subset A of X .

Proof. We may assume that A is a subset of B_X .

Step 1. $wk_X(A) \leq \iota_1(A)$.

Let us assume that $wk_X(A) > 0$ and fix an arbitrary $\epsilon \in (0, wk_X(A))$. It follows from [31, Proposition 7.1] that there exists a sequence $(x_k)_k$ in A such that

$$\int_{E_k} |x_k| d\mu > \epsilon, \quad k = 1, 2, \dots, \quad (4.4)$$

where $E_k = \{t \in \Omega : |x_k(t)| \geq k\}$.

For each $k \in \mathbf{N}$, Chebyshev's inequality gives $\mu(E_k) \leq \frac{1}{k}$ and hence the sequence $(x_k \chi_{E_k})_k$ converges to 0 in measure. By [1, Lemma 5.2.1], there exist a subsequence $(x_{k_n} \chi_{E_{k_n}})_n$ of $(x_k \chi_{E_k})_k$ and a sequence of disjoint measurable sets $(A_n)_n$ such that

$$\|x_{k_n} \chi_{E_{k_n}} - x_{k_n} \chi_{E_{k_n}} \chi_{A_n}\|_1 \rightarrow 0 \quad (n \rightarrow \infty). \quad (4.5)$$

Set $B_n = E_{k_n} \cap A_n$ and $f_n = \text{sign}(x_{k_n}) \chi_{B_n}$ for each $n \in \mathbf{N}$. Then $(f_n)_n$ is weakly 1-summable in X^* and $\|(f_n)_n\|_1^w \leq 1$. Combining (4.4) with (4.5), we get

$$\limsup_n |\langle f_n, x_{k_n} \rangle| \geq \epsilon,$$

which implies that $\iota_1(A) \geq \epsilon$. Since $\epsilon \in (0, wk_X(A))$ is arbitrary, we conclude Step 1.

Step 2. $\iota_1(A) \leq wk_X(A)$.

Similarly, we can assume that $\iota_1(A) > 0$ and fix an arbitrary $\epsilon \in (0, \iota_1(A))$. Then there exist sequences $(x_n)_n$ in A and $(f_n)_n \in l_1^w(X^*)$ with $\|(f_n)_n\|_1^w \leq 1$ such that

$$\int_{\Omega} f_n \cdot x_n d\mu > \epsilon, \quad n = 1, 2, \dots \quad (4.6)$$

It is easy to verify that

$$\|(f_n)_n\|_1^w = \sup_n \left\| \sum_{k=1}^n |f_k| \right\|.$$

Thus, we get

$$\sum_{k=1}^{\infty} |f_k| \leq 1, \quad \mu\text{-a.e.}$$

For the sake of convenience, we may assume that $\sum_{k=1}^{\infty} |f_k| \leq 1$ everywhere.

Let $\delta > 0$ be arbitrary. By (4.6), we obtain $N_1 \in \mathbf{N}$ such that

$$\int_{E_1} f_1 \cdot x_1 d\mu > \epsilon,$$

where $E_1 = \left\{ t \in \Omega : \sum_{k=N_1}^{\infty} |f_k(t)| < \frac{\delta}{2} \right\}$.

Set $\widetilde{f}_1 = f_1 \cdot \chi_{E_1}$ and $\widetilde{f}_n = f_n \cdot \chi_{E_1^c}$ ($n \geq N_1$). Then \widetilde{f}_1 is disjoint from \widetilde{f}_n for each $n \geq N_1$. Moreover, for each $n \geq N_1$, we have

$$\int_{\Omega} \widetilde{f}_n \cdot x_n \, d\mu = \int_{\Omega} f_n \cdot x_n \, d\mu - \int_{E_1} f_n \cdot x_n \, d\mu > \epsilon - \frac{\delta}{2}. \tag{4.7}$$

Obviously,

$$\sum_{k=N_1}^{\infty} |\widetilde{f}_k| \leq \sum_{k=N_1}^{\infty} |f_k| \leq 1 \text{ everywhere.}$$

By (4.7), we obtain $N_2 > N_1$ such that

$$\int_{E_2} \widetilde{f}_{N_1} \cdot x_{N_1} \, d\mu > \epsilon - \frac{\delta}{2}, \tag{4.8}$$

where $E_2 = \left\{ t \in \Omega : \sum_{k=N_2}^{\infty} |\widetilde{f}_k(t)| < \frac{\delta}{2^2} \right\}$.

Set $\widetilde{\widetilde{f}}_2 = \widetilde{f}_{N_1} \cdot \chi_{E_2}$ and $\widetilde{\widetilde{f}}_n = \widetilde{f}_n \cdot \chi_{E_2^c}$ ($n \geq N_2$). Then, by (4.8), we get

$$\int_{F_1} |x_{N_1}| \, d\mu \geq \int_{\Omega} \widetilde{\widetilde{f}}_2 \cdot x_{N_1} \, d\mu = \int_{E_2} \widetilde{f}_{N_1} \cdot x_{N_1} \, d\mu > \epsilon - \frac{\delta}{2}, \tag{4.9}$$

where F_1 is the support of $\widetilde{\widetilde{f}}_2$.

By (4.7), for each $n \geq N_2$, we have

$$\int_{\Omega} \widetilde{\widetilde{f}}_n \cdot x_n \, d\mu = \int_{E_2^c} \widetilde{f}_n \cdot x_n \, d\mu = \int_{\Omega} \widetilde{f}_n \cdot x_n \, d\mu - \int_{E_2} \widetilde{f}_n \cdot x_n \, d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}.$$

In a similar way, we obtain $N_3 > N_2$ such that

$$\int_{E_3} \widetilde{\widetilde{f}}_{N_2} \cdot x_{N_2} \, d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}, \tag{4.10}$$

where $E_3 = \left\{ t \in \Omega : \sum_{k=N_3}^{\infty} |\widetilde{\widetilde{f}}_k(t)| < \frac{\delta}{2^3} \right\}$.

Set $\widetilde{\widetilde{\widetilde{f}}}_3 = \widetilde{\widetilde{f}}_{N_2} \cdot \chi_{E_3}$ and $\widetilde{\widetilde{\widetilde{f}}}_n = \widetilde{\widetilde{f}}_n \cdot \chi_{E_3^c}$ ($n \geq N_3$). Then, by (4.10), we get

$$\int_{F_2} |x_{N_2}| \, d\mu \geq \int_{\Omega} \widetilde{\widetilde{\widetilde{f}}}_3 \cdot x_{N_2} \, d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}, \tag{4.11}$$

where F_2 is the support of $\widetilde{\widetilde{\widetilde{f}}}_3$.

Since $\widetilde{\widetilde{f}}_2$ is disjoint from \widetilde{f}_n for each $n \geq N_2$, the set F_2 is also disjoint from F_1 .

A similar computation shows that for each $n \geq N_3$,

$$\int_{\Omega} \widetilde{\widetilde{\widetilde{f}}}_n \cdot x_n \, d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2} - \frac{\delta}{2^3}.$$

Thus, there exists $N_4 > N_3$ such that

$$\int_{E_4} \widetilde{\widetilde{\widetilde{f}}}_{N_3} \cdot x_{N_3} \, d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2} - \frac{\delta}{2^3}, \tag{4.12}$$

where $E_4 = \left\{ t \in \Omega : \sum_{k=N_4}^{\infty} |\widetilde{\widetilde{\widetilde{f}}}_k(t)| < \frac{\delta}{2^4} \right\}$.

Set $\widetilde{\widetilde{f}}_4 = \widetilde{\widetilde{f}}_{N_3} \cdot \chi_{E_4}$ and $\widetilde{\widetilde{f}}_n = \widetilde{\widetilde{f}}_n \cdot \chi_{E_4^c}$ for $n \geq N_4$. The inequality (4.12) yields

$$\int_{F_3} |x_{N_3}| d\mu \geq \int_{\Omega} \widetilde{\widetilde{f}}_4 \cdot x_{N_3} d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2} - \frac{\delta}{2^3},$$

where F_3 is the support of $\widetilde{\widetilde{f}}_4$.

Since \widetilde{f}_3 is disjoint from \widetilde{f}_{N_3} , the set F_3 is disjoint from F_2 . Since \widetilde{f}_2 is disjoint from \widetilde{f}_{N_3} , the set F_3 is disjoint from F_1 .

Continuing in a similar manner, we get a subsequence $(x_{N_k})_k$ of $(x_n)_n$ and a sequence of disjoint measurable sets $(F_k)_k$ such that

$$\int_{F_k} |x_{N_k}| d\mu > \epsilon - \sum_{i=1}^k \frac{\delta}{2^i} > \epsilon - \delta, \quad k = 1, 2, \dots$$

Claim: $wk_X(A) \geq \epsilon - \delta$.

Indeed, fix any $c > 0$. Then, for each $k \in \mathbf{N}$,

$$\begin{aligned} \epsilon - \delta &< \int_{F_k} |x_{N_k}| d\mu \\ &= \int_{F_k \cap \{|x_{N_k}| \geq c\}} (|x_{N_k}| - c) d\mu + c\mu(F_k \cap \{|x_{N_k}| \geq c\}) + \int_{F_k \cap \{|x_{N_k}| < c\}} |x_{N_k}| d\mu \\ &\leq \int_{F_k \cap \{|x_{N_k}| \geq c\}} (|x_{N_k}| - c) d\mu + c\mu(F_k \cap \{|x_{N_k}| \geq c\}) + c\mu(F_k \cap \{|x_{N_k}| < c\}) \\ &\leq \int_{\{|x_{N_k}| \geq c\}} (|x_{N_k}| - c) d\mu + c\mu(F_k) \\ &\leq \sup_{x \in A} \int_{\{|x| \geq c\}} (|x| - c) d\mu + c\mu(F_k). \end{aligned}$$

Since $(F_k)_k$ is disjoint and μ is finite, $\mu(F_k) \rightarrow 0$ ($k \rightarrow \infty$).

Letting $k \rightarrow \infty$, we get

$$\epsilon - \delta \leq \sup_{x \in A} \int_{\{|x| \geq c\}} (|x| - c) d\mu.$$

Again by [31, Proposition 7.1], we prove the claim.

Since $\delta > 0$ is arbitrary, we get

$$wk_X(A) \geq \epsilon.$$

By the arbitrariness of $\epsilon \in (0, \iota_1(A))$, we obtain

$$\iota_1(A) \leq wk_X(A).$$

This completes the proof. □

Theorem 4.6. Let $X = l_1(\mathbf{N}, \mathbf{R})$. Then

$$wk_X(A) = \iota_1(A)$$

for each bounded subset A of X .

Proof. We may assume that A is a subset of B_X .

Step 1. $wk_X(A) \leq \iota_1(A)$.

We may assume that $wk_X(A) > 0$ and fix an arbitrary $c \in (0, wk_X(A))$. By [31, Lemma 7.2], we obtain two sequences $(p_n)_n, (q_n)_n$ of natural numbers with $p_n < q_n < p_{n+1} (n \in \mathbb{N})$ and a sequence $(x_n)_n$ in A such that

$$\sum_{k=p_n}^{q_n} |x_n(k)| > c, \quad n = 1, 2, \dots$$

For each $n \in \mathbb{N}$, we can find $(\alpha_n(k))_{k=p_n}^{q_n}$ with $\sup_{p_n \leq k \leq q_n} |\alpha_n(k)| = 1$ such that

$$\sum_{k=p_n}^{q_n} x_n(k)\alpha_n(k) > c. \tag{4.13}$$

For each $n \in \mathbb{N}$, we set $f_n \in l_\infty$ by

$$f_n(k) = \begin{cases} \alpha_n(k), & p_n \leq k \leq q_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(f_n)_n$ is weakly 1-summable and $\|(f_n)_n\|_1^w \leq 1$. Indeed, for each $x \in X$, we have

$$\sum_{n=1}^{\infty} |\langle f_n, x \rangle| = \sum_{n=1}^{\infty} \left| \sum_{k=p_n}^{q_n} x(k)\alpha_n(k) \right| \leq \sum_{n=1}^{\infty} \sum_{k=p_n}^{q_n} |x(k)||\alpha_n(k)| \leq \sum_{n=1}^{\infty} \sum_{k=p_n}^{q_n} |x(k)| \leq \|x\|.$$

It follows from (4.13) that

$$\sup_{x \in A} |\langle f_n, x \rangle| \geq |\langle f_n, x_n \rangle| = \sum_{k=p_n}^{q_n} x_n(k)\alpha_n(k) > c, \quad n = 1, 2, \dots$$

This implies that $\iota_1(A) \geq c$.

Since $c \in (0, wk_X(A))$ is arbitrary, we get

$$wk_X(A) \leq \iota_1(A).$$

Step 2. $\iota_1(A) \leq wk_X(A)$.

Assume that $\iota_1(A) > 0$ and fix an arbitrary $\epsilon \in (0, \iota_1(A))$. Then there exist sequences $(x_n)_n$ in A and $(f_n)_n \in l_1^w(X^*)$ with $\|(f_n)_n\|_1^w \leq 1$ such that

$$\sum_{k=1}^{\infty} f_n(k)x_n(k) > \epsilon, \quad n = 1, 2, \dots \tag{4.14}$$

It follows from $\|(f_n)_n\|_1^w \leq 1$ that

$$\sum_{n=1}^{\infty} |f_n(k)| \leq 1, \quad k = 1, 2, \dots \tag{4.15}$$

Let $\delta > 0$ be arbitrary. By (4.14) and (4.15), there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{k \in E_1} f_1(k)x_1(k) > \epsilon,$$

where $E_1 = \left\{ k : \sum_{n=N_1}^{\infty} |f_n(k)| < \frac{\delta}{2} \right\}$.

Set $\tilde{f}_1 = f_1 \cdot \chi_{E_1}$ and $\tilde{f}_n = f_n \cdot \chi_{E_1^c} (n \geq N_1)$. By (4.14) and the definition of E_1 , we have

$$\langle \tilde{f}_n, x_n \rangle = \sum_{k=1}^{\infty} f_n(k)x_n(k) - \sum_{k \in E_1} f_n(k)x_n(k) > \epsilon - \frac{\delta}{2}, \tag{4.16}$$

for each $n \geq N_1$.

The inequality (4.15) yields that

$$\sum_{n=N_1}^{\infty} |\tilde{f}_n(k)| \leq \sum_{n=N_1}^{\infty} |f_n(k)| \leq 1, \quad k = 1, 2, \dots \tag{4.17}$$

Combining (4.16) with (4.17), we obtain $N_2 > N_1$ such that

$$\sum_{k \in E_2} \tilde{f}_{N_1}(k)x_{N_1}(k) > \epsilon - \frac{\delta}{2}, \tag{4.18}$$

where $E_2 = \left\{ k : \sum_{n=N_2}^{\infty} |\tilde{f}_n(k)| < \frac{\delta}{2^2} \right\}$.

Set $\tilde{\tilde{f}}_2 = \tilde{f}_{N_1} \cdot \chi_{E_2}$ and $\tilde{\tilde{f}}_n = \tilde{f}_n \cdot \chi_{E_2^c}$ ($n \geq N_2$). Then, by (4.18), we get

$$\sum_{k \in F_1} |x_{N_1}(k)| \geq \sum_{k=1}^{\infty} \tilde{\tilde{f}}_2(k)x_{N_1}(k) > \epsilon - \frac{\delta}{2},$$

where F_1 is the support of $\tilde{\tilde{f}}_2$.

Moreover, by (4.16) and the definition of E_2 , we get

$$\left\langle \tilde{\tilde{f}}_n, x_n \right\rangle = \sum_{k=1}^{\infty} \tilde{f}_n(k)x_n(k) - \sum_{k \in E_2} \tilde{f}_n(k)x_n(k) > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}, \tag{4.19}$$

for each $n \geq N_2$.

By (4.19), there exists $N_3 > N_2$ such that

$$\sum_{k \in E_3} \tilde{\tilde{f}}_{N_2}(k)x_{N_2}(k) > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}, \tag{4.20}$$

where $E_3 = \left\{ k : \sum_{n=N_3}^{\infty} |\tilde{\tilde{f}}_n(k)| < \frac{\delta}{2^3} \right\}$.

Set $\tilde{\tilde{\tilde{f}}}_3 = \tilde{\tilde{f}}_{N_2} \cdot \chi_{E_3}$ and $\tilde{\tilde{\tilde{f}}}_n = \tilde{\tilde{f}}_n \cdot \chi_{E_3^c}$ ($n \geq N_3$). Then, by (4.20), we get

$$\sum_{k \in F_2} |x_{N_2}(k)| \geq \sum_{k=1}^{\infty} \tilde{\tilde{\tilde{f}}}_3(k)x_{N_2}(k) > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2},$$

where F_2 is the support of $\tilde{\tilde{\tilde{f}}}_3$.

Since $\tilde{\tilde{f}}_2$ is disjoint from $\tilde{\tilde{f}}_{N_2}$, the set F_2 is disjoint from F_1 .

As in the proof of Theorem 4.5, we get a subsequence $(x_{N_k})_k$ of $(x_n)_n$ and a sequence $(F_k)_k$ of pairwise disjoint subsets of \mathbb{N} such that

$$\sum_{i \in F_k} |x_{N_k}(i)| > \epsilon - \sum_{i=1}^k \frac{\delta}{2^i} > \epsilon - \delta, \quad k = 1, 2, \dots$$

Finally, we claim: $wk_X(A) \geq \epsilon - \delta$.

Let us fix $n \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, we have

$$\epsilon - \delta < \sum_{i=1}^{\infty} |x_{N_k}(i)|\chi_{F_k}(i) = \sum_{i=1}^n |x_{N_k}(i)|\chi_{F_k}(i) + \sum_{i=n+1}^{\infty} |x_{N_k}(i)|\chi_{F_k}(i) \leq \sum_{i=1}^n \chi_{F_k}(i) + \sup_{x \in A} \sum_{i=n+1}^{\infty} |x(i)|.$$

Since $(F_k)_k$ is disjoint pairwise, $\sum_{i=1}^n \chi_{F_k}(i) = 0$ for k large enough.

This implies

$$\epsilon - \delta \leq \inf_n \sup_{x \in A} \sum_{i=n+1}^{\infty} |x(i)|.$$

It follows from [31, Proposition 7.3] that

$$\epsilon - \delta \leq wk_X(A).$$

Since $\delta > 0$ is arbitrary, we get

$$\epsilon \leq wk_X(A).$$

The arbitrariness of $\epsilon \in (0, \iota_1(A))$ concludes the proof. \square

Definition 4.7. Let $1 \leq p \leq \infty$. We say that a Banach space X has quantitative Pełczyński's property (V) of order p (property p -(V) $_q$ in short) with a constant $C > 0$ if for every Banach space Y and every operator $T : X \rightarrow Y$, one has

$$wk_{X^*}(T^*) \leq C \cdot uc_p(T).$$

If a Banach space X has property p -(V) $_q$ with some constant $C > 0$, we say that X has property p -(V) $_q$.

Clearly, if a Banach space X has property p -(V) $_q$, then it has property p -(V).

Let $1 \leq p \leq \infty$ and let X be a Banach space. For a bounded subset A of X^* , we set

$$\eta_p(A) = \sup \left\{ \limsup_n \sup_{x^* \in A} |\langle x^*, x_n \rangle| : (x_n)_n \in l_p^w(X), \|(x_n)_n\|_p^w \leq 1 \right\}.$$

Then $\eta_p(A) = 0$ if and only if A is a p -(V) set. Note that the quantity $\eta_1(\cdot)$ is equal to the quantity $\eta(\cdot)$ introduced in [36].

The proof of the following theorem is similar to that of Theorem 4.3.

Theorem 4.8. Let X be a Banach space and let $1 \leq p < \infty$. The following statements are equivalent:

- (1) X has property p -(V) $_q$;
- (2) there exists a constant $C > 0$ such that for each bounded subset A of X^* , one has

$$wk_{X^*}(A) \leq C \cdot \eta_p(A).$$

Theorem 4.9. Let $1 \leq p \leq \infty$. If a Banach space X has property p -(V) $_q$ with a constant C , then every quotient of X has property p -(V) $_q$ with $2C$.

Proof. Let M be a closed subspace of X . Suppose that X has property p -(V) $_q$ with a constant $C > 0$. We'll show that the quotient X/M has property p -(V) $_q$ with $2C$. Fix a Banach space Y and an operator $S : X/M \rightarrow Y$. Then we have

$$wk_{X^*}(Q^*S^*) \leq C \cdot uc_p(SQ),$$

where $Q : X \rightarrow X/M$ is the canonical quotient map. By (4.3), we get

$$wk_{(X/M)^*}(S^*) \leq 2wk_{X^*}(Q^*S^*) \leq 2C \cdot uc_p(SQ) \leq 2C \cdot uc_p(S),$$

which completes the proof. \square

Theorem 4.10. Let X be a Banach space and let $1 \leq p \leq \infty$. Then:

- (1) If X has property p -(V) $_q$ with a constant C , then X^* has property p -(V) $_q^*$ with the same constant C ;
- (2) If X^* has property p -(V) $_q^*$ with a constant C , then X has property p -(V) $_q$ with $2C$.

Proof. (1) Let Z be a Banach space and let S be an operator from Z into X^* . Applying the assumption to S^*J_X , we get

$$wk_{X^*}(J_X^*S^{**}) \leq C \cdot uc_p(S^*J_X).$$

This yields

$$wk_{X^*}(S) = wk_{X^*}(J_X^* S^{**} J_Z) \leq wk_{X^*}(J_X^* S^{**}) \leq C \cdot uc_p(S^* J_X) \leq C \cdot uc_p(S^*),$$

which completes the proof of (1).

The assertion (2) follows immediately from (1) and Theorem 4.4. \square

Let us mention that the converse of Theorem 4.10 is false. Indeed, $X = \left(\sum_{n=1}^{\infty} \oplus l_{\infty}^n\right)_1$ enjoys property $1-(V^*)_q$ with constant 1 as following from the following Theorem 4.11. But X^* fails property $p-(V)$ for each $1 \leq p < \infty$ because X^* contains a 1-complemented subspace isometric to l_1 (see [38, p. 73] for this fact) and l_1 fails property $p-(V)$ for each $1 \leq p < \infty$ by Theorem 2.3 and Theorem 2.4.

Theorem 4.11. *Let $X = \left(\sum_{\gamma \in \Gamma} \oplus X_{\gamma}\right)_1$, where each X_{γ} is reflexive. Then*

$$wk_X(A) \leq \iota_1(A)$$

for each bounded subset A of X .

Proof. *If Γ is finite, then X is reflexive and the conclusion is obvious. We may suppose that Γ is infinite. Let A be a bounded subset of X . We may assume that $wk_X(A) > 0$ and fix an arbitrary $c \in (0, wk_X(A))$. By [31, Lemma 7.2], we obtain, by induction, a sequence $(F_n)_n$ of pairwise disjoint finite subsets of Γ and a sequence $(x_n)_n$ in A such that*

$$\sum_{\gamma \in F_n} \|x_n(\gamma)\| > c, \quad n = 1, 2, \dots$$

For each $n \in \mathbf{N}$, there exists $(f_n(\gamma))_{\gamma \in F_n} \in \left(\sum_{\gamma \in F_n} \oplus X_{\gamma}^*\right)_{\infty}$ with $\max_{\gamma \in F_n} \|f_n(\gamma)\| = 1$ such that

$$\sum_{\gamma \in F_n} \langle f_n(\gamma), x_n(\gamma) \rangle > c.$$

For each $n \in \mathbf{N}$, define $\varphi_n \in \left(\sum_{\gamma \in \Gamma} \oplus X_{\gamma}^*\right)_{\infty}$ by

$$\varphi_n(\gamma) = \begin{cases} f_n(\gamma), & \gamma \in F_n, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(F_n)_n$ is pairwise disjoint, it is easy to verify that $(\varphi_n)_n$ is weakly 1-summable and $\|(\varphi_n)_n\|_1^w \leq 1$. Moreover, for each $n \in \mathbf{N}$,

$$\sup_{x \in A} |\langle \varphi_n, x \rangle| \geq |\langle \varphi_n, x_n \rangle| = \sum_{\gamma \in F_n} \langle f_n(\gamma), x_n(\gamma) \rangle > c.$$

This yields that $\iota_1(A) \geq c$. Since $c \in (0, wk_X(A))$ is arbitrary, we get the conclusion. \square

Theorem 4.12. *Let $X = c_0(\mathbf{N}, \mathbf{R})$. Then*

$$wk_X(A) = \eta_1(A)$$

for each bounded subset A of X^* .

The proof is essentially analogous to Theorem 4.6, only interchanging the role of X and X^* .

H. Krulišová proved in [36] that $C_0(\Omega)$ has property $1-(V)_q$ with constant π (constant 2 in the real case) for every locally compact space Ω .

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