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Positive *p*-summing operators and disjoint *p*-summing operators

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Abstract

In the present paper, we introduce a new concept of positive p-majorizing operators as a dual notion of positive p-summing operators and generalize the concept of majorizing operators introduced by Schaefer (Isr J Math 13:400–415, 1972). We introduce the concept of positive (p, q)-dominated operators and prove a positive version of the famous Kwapień's factorization theorem for (p, q)-dominated operators via positive p-majorizing operators. We also introduce the notion of disjoint p-summing operators which is a new larger class of operators than positive p-summing operators and use it to characterize the Radon–Nikodým property. Finally, we investigate the maximal properties of these four classes of operators and prove that they are maximal in corresponding sense.

Keywords Positive p-summing operators · Positive p-majorizing operators · Positive (p, q)-dominated operators · Disjoint p-summing operators

Mathematics Subject Classification Primary 47B10 · 46B28 · 46B42 · 46B45

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1 Introduction

Abstract M-spaces (briefly, AM-spaces) and abstract L-spaces (briefly, AL-spaces) are the most important examples of Banach lattices. In 1971, Schlotterbeck [21] (see also [20]) presented characterizations of AM- and AL-spaces which are quite different from the classical Kakutani's representation theorems for AM-spaces with a unit and AL-spaces: A Banach lattice X is isometric lattice isomorphic to an AL-space (AM-space, respectively) if and only if every positive unconditionally summable sequence in X is absolutely summable (resp. every norm null sequence in X is order bounded), that is, the identity map I_X on X takes positive unconditionally summable sequences in X to absolutely summable sequences (resp. I_X takes norm null sequences in X to order bounded sequences). In 1972, Schaefer [19] generalized these properties of the identity map on AL- and AM-spaces in a natural way and introduced the concept of *cone absolutely summing* operators and *majorizing* operators, respectively. Furthermore, Schaefer [19] characterized cone absolutely summing operators (majorizing operators, respectively) by factoring positively through AL-spaces (resp. AM-spaces). On the other hand, by introducing the l-norm on the class of all cone absolutely summing operators (resp. the m-norm on the class of all majorizing operators), Schaefer [19] extended Schlotterbeck's characterizations of AL-spaces (resp. AM-spaces). Cone absolutely summing operators are also called *order summing* [9]. In [9], the relationships among order summing operators on $L_p(\mu)$, the Radon–Nikodým property and compact operators are studied. It was shown [9, Theorem 8,p.110] and [9, Notes and remarks,p.119] that a Banach space E has the Radon–Nikodým property if and only if every order summing operator $T : L_p(\mu) \to E$ is representable by a function f in $L_{p^*}(\mu, E)(1 < p < \infty)$. Consequently, every order summing operator $T : L_p(\mu) \to E$ is compact whenever E has the Radon–Nikodým property. In 1971, Krsteva [13] (written in Russian) (see also [10]) extended the concept of cone absolutely summing operators and introduced the concept of *latticially p-summing* operators. It was shown [10] that an operator T is latticially p-summing if and only if T^{**} is. Being unaware of [10,13], Blasco [3] introduced the concept of *positive p-summing* operators from $L_{p^*}(\mathbf{T})$($\mathbf{T}$ denotes the torus)($\frac{1}{p} + \frac{1}{p^*} = 1$) to a Banach space E and interpreted the space of boundary values of harmonic E-valued functions in $h^p_E(D)$ as the space of all positive p-summing operators from $L_{p^*}(\mathbf{T})$ to E. Subsequently, Blasco [4] formally introduced the concept of positive p-summing operators which is exactly the same as latticially p-summing operators. It was shown [4] that every positive p-summing operator from $L_{p^*}(\mu)$ to any Banach space E is cone absolutely summing. Blasco [4] generalized, strengthened [9, Theorem 8,p.110] and characterized the Radon–Nikodým property in terms of positive p-summing operators: A Banach space E has the Radon–Nikodým property if and only if every positive p^*-summing operator $T : L_p(\mu) \to E$ is representable by a function $f \in L_{p^*}(\mu, E)$. Therefore, every positive p^*-summing operator $T : L_p(\mu) \to E$ is compact provided that E has the Radon–Nikodým property. In 1998, Zhukova [22] investigated some simple properties of latticially p-summing operators and established the Pietsch domination theorem for latticially p-summing operators. Blasco [5] generalized the concept of positive p-summing operators to positive (p, q)-summing operators and related this class of operators to (p, q)-concave operators. Achour and Belacel [1] introduced the notion of positive strongly (p, q)-summing operators and presented the Pietsch domination/factorization theorems for positive strongly p-summing operators. Furthermore, the duality relations between positive p-summing operators and positive strongly p^*-summing operators are proved [1].

The aim of this paper is to continue to develop the theory of positive p-summing operators. The present paper is organized as follows.

Section 2 begins with a new characterization of majorizing operators. Based on this characterization, we extend the notion of majorizing operators to more general case and introduce the concept of *positive p-majorizing* operators. Furthermore, we generalize the duality relationships between cone absolutely summing operators and majorizing operators ([19, Propositon 3]) and prove the duality relations between positive p-summing operators and positive p^*-majorizing operators. It should be mentioned that Schaefer [19] established the duality relationships between cone absolutely summing operators and majorizing operators by means of the well-known duality between AL- and AM-spaces. However, we establish the duality relations between positive p-summing operators and positive p^*-majorizing operators by using the principle of local reflexivity in Banach lattices together with the principle of local reflexivity in Banach spaces. Hence our technique seems to be new and different. By the end of this section, we describe positive p-summing operators and positive p-majorizing operators in terms of tensor products equipped with suitable reasonable cross norms. Labuschagne [15] constructed isometric preduals of the space of positive p-summing operators acting between Banach lattices and conjugate Banach spaces via $|\epsilon|$-norm and ${}^tM$-norm. In this section, we introduce positive analogues $|d|_p$, $|g|_p$ of the Chevet-Saphar norms d_p, g_p and use them to construct isomorphic preduals of the space of positive p-summing operators acting between Banach lattices and general Banach spaces (resp. the space of positive p-majorizing operators).

The definition of (p, q)-dominated operators goes back to Kwapień [14] ([7,18]) who proved the famous factorization theorem for (p, q)-dominated operators. In Sect. 3, we introduce the concept of *positive (p, q)-dominated* operators that are positive analogues of (p, q)-dominated operators. By using a function space introduced by Achour and Belacel [1] and the concept of positive p-majorizing operators introduced in Sect. 2, we establish a positive version of Kwapień's factorization theorem for positive (p, q)-dominated operators. Our method seems to be new. As an application of this factorization theorem, we prove that an operator T is positive (p, q)-dominated if and only if T^* is positive (q, p)-dominated if and only if T^{**} is positive (p, q)-dominated. In this section, we also introduce a positive analogue $|\alpha|_{p,q}$ of the Lapresté norm $\alpha_{p,q}$ and use it to construct an isomorphic predual of the space of positive (p, q)-dominated operators.

Section 4 deals with a new larger class of operators than positive p-summing operators: *disjoint p-summing* operators. The main result of this section is Theorem 4.5, which states that every disjoint p-summing operator from $L_{p^*}(\mu)$ to any Banach space E is also cone absolutely summing. As an application, we characterize the Radon–Nikodým property in terms of disjoint p-summing operators. Another application of Theorem 4.5 is to characterize disjoint p-summing operators in terms of cone absolutely summing operators and disjointness preserving operators. Finally, we prove that an operator T is disjoint p-summing if and only if T^{**} is.

The final section is devoted to investigating the maximal properties of the various classes of operators introduced in the paper. It is well-known that minimality and maximality are two extremely important properties in the theory of Banach operator ideals. With respect to a natural partial order in the family of Banach operator ideals, a maximal Banach operator ideal $(\mathfrak{A}, \mathbf{A})$ is the largest of all the Banach operator ideals which coincide with $(\mathfrak{A}, \mathbf{A})$ for finite dimensional spaces. The most important ingredient in Banach operator ideals is the ideal property, that is, two-sided compositions by operators. However, the classes of operators investigated in this paper are less general than Banach operator ideals in that one-sided compositions by positive operators, even by disjointness preserving operators, are required. Therefore, the maximality properties of the classes of operators at the present paper have to be dealt with differently. In this section, we first introduce the concept of the maximal hull $(\Lambda_p^{\max}, \|\cdot\|_{\Lambda_p}^{\max})$ of the class $(\Lambda_p, \|\cdot\|_{\Lambda_p})$ of positive p-summing operators by finite dimensional sublattices rather than finite dimensional subspaces. We not only characterize the maximal hull $(\Lambda_p^{\max}, \|\cdot\|_{\Lambda_p}^{\max})$ by means of the right side composition by positive finite rank operators, but also prove that the class of positive p-summing operators is maximal under the condition of order completeness. As for the class $(\Gamma_p, \|\cdot\|_{\Gamma_p})$ of disjoint p-summing operators, we also introduce the maximal hull $(\Gamma_p^{\max}, \|\cdot\|_{\Gamma_p}^{\max})$ by finite dimensional sublattices and characterize it in terms of disjointness preserving finite rank operators. It is proved that the class of disjoint p-summing operators is maximal. Since the maximal hull of Banach operator ideals can be restated by finite rank operators [18, Theorem 8.7.4], we introduce the maximal hull of the class of positive p-majorizing operators by means of the left side composition by positive finite rank operators and show that the class of positive p-majorizing operators is maximal. Finally, we introduce the maximal hull of the class of positive (p, q)-dominated operators in terms of the two-sided composition by positive finite rank operators and prove that the class of positive (p, q)-dominated operators is maximal under the hypothesis of order completeness.

1.1 Notation and preliminary

Our notation and terminology are standard as may be found in [17,18]. Throughout the paper, X, Y, Z will always denote real Banach lattices, whereas E, F, G will denote real Banach spaces. By an operator, we always mean a bounded linear operator. For a Banach lattice X, we denote by X_+ the positive cone of X, i.e., $X_+ := \{x \in X : x \geq 0\}$. Two elements $x_1, x_2 \in X$ are called *disjoint*(denoted by $x_1 \perp x_2$) if $|x_1| \wedge |x_2| = 0$. We write $LDim(X)$ for the collection of all finite dimensional sublattices of X. If N is a closed subspace of E, we denote by i_N the canonical inclusion from N into E and by Q_N the natural quotient map from E onto E/N. We let $N^\perp := \{u^* \in E^* : \langle u^*, u\rangle = 0$ for all $u \in N\}$. We write $FIN(E)$ for the collection of all finite-dimensional subspaces of E and $COFIN(E)$ for the collection of all finite co-dimensional subspaces of E. An operator $T : X \to Y$ which preserves the lattice operations is called *lattice homomorphism*, that is, $T(x_1 \vee x_2) = Tx_1 \vee Tx_2$ for all $x_1, x_2 \in X$. An one-to-one, surjective lattice homomorphism is called *lattice isomorphism*. As customary, B_E denotes the closed unit ball of E, E^* its linear dual and I_E the identity map on

E. We denote by $\mathcal{L}(E, F)$ (resp. $\mathcal{F}(E, F)$) the space of all operators (resp. finite rank operators) from E to F. For Banach lattices X and Y, $\mathcal{F}(X, Y)_+$ stands for the set of all positive finite rank operators from X to Y. The letters p, q will designate elements of $[1, +\infty]$, and p^* denotes the exponent conjugate to p (i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$). For $u_1, u_2, \ldots, u_n$ in E, we let

$$\left\|(u_i)_{i=1}^n\right\|_p := \left(\sum_{i=1}^n \|u_i\|^p\right)^{\frac{1}{p}}.$$

A sequence $(u_n)_n$ in E is called *weakly p-summable* if the scalar sequence $(\langle u^*, u_n\rangle)_n$ is in l_p for every $u^* \in E^*$. We denote by $l_p^w(E)$ the space of all weakly p-summable sequences in E, endowed with the norm

$$\|(u_n)_n\|_p^w := \sup_{u^* \in B_{E^*}} \left(\sum_{n=1}^\infty |\langle u^*, u_n\rangle|^p\right)^{\frac{1}{p}}.$$

It is a well-known result of Grothendieck ([11], [8, Proposition 2.2]) that the canonical correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{L}(l_{p^*}, E)$ onto $l_p^w(E)$ when $1 < p < \infty$; For $p = 1$, the isometric isomorphism is from $\mathcal{L}(c_0, E)$ onto $l_1^w(E)$; For $p = \infty$, the isometric isomorphism is from $\mathcal{L}(l_1, E)$ onto $l_\infty(E)$.

Consider the case where Banach space E is replaced by a Banach lattice X. Let $(x_n)_n \in l_p^w(X)$ and each $x_n \in X_+$. It follows from the basic inequality $|\langle x^*, x\rangle| \leq \langle |x^*|, |x|\rangle (x^* \in X^*, x \in X)$ that

$$\|(x_n)_n\|_p^w = \sup_{x^* \in (B_{X^*})_+} \left(\sum_{n=1}^\infty \langle x^*, x_n\rangle^p\right)^{\frac{1}{p}}, \tag{1.1}$$

where $(B_{X^*})_+ = \{x^* \in B_{X^*} : x^* \geq 0\}$.

Given positive elements $x_1, x_2, \ldots, x_n$ in X, a simple use of the usual duality between l_p^n and $l_{p^*}^n$ yields the following useful equality:

$$\left\|(x_i)_{i=1}^n\right\|_p^w = \sup_{a \in \left(B_{l_{p^*}^n}\right)_+} \left\|\sum_{i=1}^n a_i x_i\right\|,$$

where $(B_{l_{p^*}^n})_+ = \{a = (a_i)_{i=1}^n \in B_{l_{p^*}^n} : a_i \geq 0, i = 1, 2, \ldots, n\}$.

The reader is referred to [8,17,18] for any unexplained notation or terminology.

2 Positive *p*-summing operators and positive *p*-majorizing operators

Definition 2.1 [19] An operator $T : X \to E$ is called *cone absolutely summing*(*c.a.s* for short) if for every positive unconditionally summable sequences $(x_n)_n$ in X, the sequence $(Tx_n)_n$ is absolutely summable in E.

Schaefer ([20, Lemma 3.2,Proposition 3.3] and [19, Proposition 1]) characterized $c.a.s$ operators by various equivalences, in particular, an operator $T : X \to E$ is $c.a.s$ if and only if there exists a constant $C > 0$ such that

$$\sum_{i=1}^{n} \|Tx_i\| \leq C \sup_{x^* \in B_{X^*}} \sum_{i=1}^{n} |\langle x^*, x_i \rangle|,$$

for all finite families $(x_i)_{i=1}^n$ in X_+.

Definition 2.2 [19] An operator $S : E \to X$ is called *majorizing* if for every norm null sequence $(u_n)_n$ in E, the sequence $(Su_n)_n$ is order bounded.

From the well-known duality between AL-spaces and AM-spaces and from the factoring characterizations given in [19, Proposition 1,Proposition 2], Schaefer [19] deduced the duality relationship between $c.a.s$ operators and majorizing operators as follows.

Theorem 2.3 [19] *Let X, Y be Banach lattices and E, F be Banach spaces. Then*

(a) *An operator $T : X \to E$ is c.a.s if and only if T^* is majorizing.*
(b) *An operator $S : F \to Y$ is majorizing if and only if S^* is c.a.s.*

We give a new characterization of majorizing operators as follows. Although this characterization is a reformulation of Theorem 2.3, we present a proof here for the sake of completeness.

Theorem 2.4 *The following statements are equivalent for an operator $S : E \to X$:*

(a) *S is majorizing.*
(b) *There exists a constant $C > 0$ such that*

$$\sum_{j=1}^{n} |\langle x_j^*, Su_j \rangle| \leq C \sup_{x^{**} \in B_{X^{**}}} \sum_{j=1}^{n} |\langle x^{**}, x_j^* \rangle|,$$

for all finite families $(u_j)_{j=1}^n$ in B_E and $(x_j^*)_{j=1}^n$ in $(X^*)_+$.

Proof $(a) \Rightarrow (b)$. Suppose that S is majorizing. It follows from [20, Proposition 3.4] that there exist a constant $C > 0$ and $x_0^{**} \in (X^{**})_+$ with $\|x_0^{**}\| \leq C$ such that

$SB_E \subseteq [-x_0^{**}, x_0^{**}]$. For any finite families $(u_j)_{j=1}^n$ in B_E and $(x_j^*)_{j=1}^n$ in $(X^*)_+$, we get

$$\sum_{j=1}^n |\langle x_j^*, Su_j\rangle| \leq \sum_{j=1}^n \langle x_j^*, |Su_j|\rangle \leq \sum_{j=1}^n \langle x_0^{**}, x_j^*\rangle \leq C \sup_{x^{**}\in B_{X^{**}}} \sum_{j=1}^n |\langle x^{**}, x_j^*\rangle|.$$

$(b) \Rightarrow (a)$. By Theorem 2.3(b), it suffices to show that S^* is $c.a.s.$

Given any finite sequence $(x_j^*)_{j=1}^n$ in $(X^*)_+$. For each $1 \leq j \leq n$, we choose $u_j^{**} \in B_{E^{**}}$ with $\|S^*x_j^*\| = \langle u_j^{**}, S^*x_j^*\rangle$. Let $M = \text{span}\{u_j^{**} : 1 \leq j \leq n\}$, $N = \text{span}\{S^*x_j^* : 1 \leq j \leq n\}$ and $\epsilon > 0$. It follows from the principle of local reflexivity in Banach spaces that there exists an operator $R : M \to E$ such that

(i) $R|_{M\cap E} = I_{M\cap E}$;
(ii) $(1-\epsilon)\|u^{**}\| \leq \|Ru^{**}\| \leq (1+\epsilon)\|u^{**}\|$, $u^{**} \in M$;
(iii) $\langle u^*, Ru^{**}\rangle = \langle u^{**}, u^*\rangle$, $u^* \in N$, $u^{**} \in M$.

Consequently,

$$\begin{aligned}\sum_{j=1}^n \|S^*x_j^*\| &= \sum_{j=1}^n |\langle u_j^{**}, S^*x_j^*\rangle| \\ &= \sum_{j=1}^n |\langle x_j^*, SRu_j^{**}\rangle| \leq C(1+\epsilon) \sup_{x^{**}\in B_{X^{**}}} \sum_{j=1}^n |\langle x^{**}, x_j^*\rangle|.\end{aligned}$$

Letting $\epsilon \to 0$, we finish the proof.

□

Theorem 2.4 leads to the following natural generalization of majorizing operators.

Definition 2.5 We say that an operator $S : E \to X$ is *positive p-majorizing* if there exists a constant $C > 0$ such that

$$\left(\sum_{j=1}^n |\langle x_j^*, Su_j\rangle|^{p^*}\right)^{\frac{1}{p^*}} \leq C\|(x_j^*)_{j=1}^n\|_{p^*}^w, \tag{2.1}$$

for all finite families $(u_j)_{j=1}^n$ in B_E and $(x_j^*)_{j=1}^n$ in $(X^*)_+$.

We denote by $\Upsilon_p(E, X)$ the space of all positive p-majorizing operators from E to X. It is easy to see that $\Upsilon_p(E, X)$ becomes a Banach space with the norm $\|\cdot\|_{\Upsilon_p}$ given by the infimum of the constants satisfying (2.1). Obviously, $\Upsilon_1(E, X) = \mathcal{L}(E, X)$ and $\|S\|_{\Upsilon_1} \leq \|S\| \leq 2\|S\|_{\Upsilon_1}$.

The following easy lemma will be used at the end of this section and its proof is straightforward.

Lemma 2.6 *An operator $S : E \to X$ is positive p-majorizing if and only if there exists a constant $K > 0$ so that*

$$\left(\sum_{j=1}^{n} \left|\left\langle x_j^*, Su_j\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \leq K \left\| \left(\left|x_j^*\right|\right)_{j=1}^{n} \right\|_{p^*}^{w},$$

for all finite choices $(u_j)_{j=1}^n$ in B_E and $(x_j^*)_{j=1}^n$ in X^*.

Definition 2.7 [4] An operator $T : X \to E$ is said to be *positive p-summing* if there exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \leq C\|(x_i)_{i=1}^{n}\|_p^w. \tag{2.2}$$

for any choice of finitely many vectors $x_1, x_2, \ldots, x_n$ in X_+.

The space of all positive p-summing operators from X to E is denoted by $\Lambda_p(X, E)$. This space becomes a Banach space with the norm $\|\cdot\|_{\Lambda_p}$ given by the infimum of the constants satisfying (2.2). For $p = \infty$, $\Lambda_\infty(X, E) = \mathcal{L}(X, E)$ and $\|T\|_{\Lambda_\infty} \leq \|T\| \leq 2\|T\|_{\Lambda_\infty}$. A standard argument shows that $\Lambda_p \subseteq \Lambda_q$ whenever $p < q$.

To extend Theorem 2.3, we need the principle of local reflexivity in Banach lattices due to Conroy and Moore [6] and Bernau [2], which plays a crucial role in Banach lattice theory.

Theorem 2.8 [2, Theorem 2] *Let X be a Banach lattice and let M be a finite-dimensional sublattice of X^{**}. Then for every finite-dimensional subspace N of X^* and every $\epsilon > 0$, there exists a lattice isomorphism R from M into X such that*

(i) $\|R\|, \|R^{-1}\| \leq 1 + \epsilon$;
(ii) $|\langle x^{**}, x^*\rangle - \langle x^*, Rx^{**}\rangle| \leq \epsilon\|x^{**}\|\|x^*\|$, *for all $x^{**} \in M$ and $x^* \in N$.*

We also need a lemma due to Lissitsin and Oja [16] that demonstrates the connection between finite-dimensional subspaces and finite-dimensional sublattices in order complete Banach lattices. This lemma will be used frequently throughout this paper.

Lemma 2.9 [16, Lemma 5.5] *Let N be a finite-dimensional subspace of an order complete Banach lattice X and let $\epsilon > 0$. Then there exist a sublattice Z of X containing N, a finite-dimensional sublattice G of Z, and a positive projection P from Z onto G such that $\|Px - x\| \leq \epsilon\|x\|$ for all $x \in N$.*

Now we are in a position to give the main result of this section.

Theorem 2.10 *An operator $T : X \to E$ is positive p-summing if and only if T^* is positive p^*-majorizing. In this case, $\|T\|_{\Lambda_p} = \|T^*\|_{\Upsilon_{p^*}}$.*

Proof Suppose that $T : X \to E$ is positive p-summing with constant C as stated in Definition 2.7. Given any finite sequences $(u_j^*)_{j=1}^n$ in B_{E^*} and $(x_j^{**})_{j=1}^n$ in $(X^{**})_+$.

Let $M = \text{span}\{x_j^{**} : 1 \leq j \leq n\}$ and $N = \text{span}\{T^* u_j^* : 1 \leq j \leq n\}$. Let $\epsilon > 0$. It follows from Lemma 2.9 that there exist a sublattice Z of X^{**} containing M, a finite-dimensional sublattice G of Z and a positive projection P from Z onto G such that

$$\|Px^{**} - x^{**}\| \leq \epsilon \|x^{**}\|, \quad x^{**} \in M. \tag{2.3}$$

By Theorem 2.8, we get a lattice isomorphism R from G into X such that $\|R\|, \|R^{-1}\| \leq 1 + \epsilon$ and

$$|\langle x^*, Rx^{**}\rangle - \langle x^{**}, x^*\rangle| \leq \epsilon \|x^{**}\| \|x^*\|, \quad x^{**} \in G, x^* \in N. \tag{2.4}$$

For each $j = 1, 2, \ldots, n$, we set $x_j = RPx_j^{**} \geq 0$. Combining (2.3) and (2.4), we get

$$\begin{aligned}
|\langle x_j^{**}, T^* u_j^*\rangle - \langle u_j^*, Tx_j\rangle| &= |\langle x_j^{**}, T^* u_j^*\rangle - \langle T^* u_j^*, RPx_j^{**}\rangle| \\
&\leq |\langle x_j^{**}, T^* u_j^*\rangle - \langle Px_j^{**}, T^* u_j^*\rangle| + |\langle Px_j^{**}, T^* u_j^*\rangle \\
&\quad - \langle T^* u_j^*, RPx_j^{**}\rangle| \\
&\leq \epsilon \|x_j^{**}\| \|T^* u_j^*\| + \epsilon \|Px_j^{**}\| \|T^* u_j^*\| \\
&\leq \epsilon \|x_j^{**}\| \|T^* u_j^*\| + \epsilon(1+\epsilon) \|x_j^{**}\| \|T^* u_j^*\| \\
&= \epsilon(2+\epsilon) \|x_j^{**}\| \|T^* u_j^*\|
\end{aligned} \tag{2.5}$$

By (2.5) and (2.3), we get

$$\begin{aligned}
\left(\sum_{j=1}^n |\langle x_j^{**}, T^* u_j^*\rangle|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{j=1}^n |\langle x_j^{**}, T^* u_j^*\rangle - \langle u_j^*, Tx_j\rangle|^p\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{j=1}^n |\langle u_j^*, Tx_j\rangle|^p\right)^{\frac{1}{p}} \\
&\leq \epsilon(2+\epsilon) \left(\sum_{j=1}^n \|x_j^{**}\|^p \|T^* u_j^*\|^p\right)^{\frac{1}{p}} + C\|(x_j)_{j=1}^n\|_p^w \\
&\leq \epsilon(2+\epsilon) \left(\sum_{j=1}^n \|x_j^{**}\|^p \|T^* u_j^*\|^p\right)^{\frac{1}{p}} \\
&\quad + C(1+\epsilon)^2 \|(x_j^{**})_{j=1}^n\|_p^w
\end{aligned}$$

Letting $\epsilon \to 0$, we get

$$\left(\sum_{j=1}^{n} |\langle x_j^{**}, T^* u_j^*\rangle|^p\right)^{\frac{1}{p}} \leq C\|(x_j^{**})_{j=1}^{n}\|_p^w.$$

This means that T^* is positive p^*-majorizing and $\|T^*\|_{\Upsilon_{p^*}} \leq \|T\|_{\Lambda_p}$.

Conversely, assume that T^* is positive p^*-majorizing with constant C as describe in Definition 2.5. Let $(x_j)_{j=1}^n$ be a finite sequence in X_+. For each $j = 1, 2, \ldots, n$, we choose $u_j^* \in B_{E^*}$ such that $\|Tx_j\| = \langle u_j^*, Tx_j\rangle$. Then, we have

$$\left(\sum_{j=1}^{n} \|Tx_j\|^p\right)^{\frac{1}{p}} = \left(\sum_{j=1}^{n} |\langle u_j^*, Tx_j\rangle|^p\right)^{\frac{1}{p}} = \left(\sum_{j=1}^{n} |\langle J_X x_j, T^* u_j^*\rangle|^p\right)^{\frac{1}{p}}$$
$$\leq C\|(x_j)_{j=1}^n\|_p^w.$$

This implies that T is positive p-summing and $\|T\|_{\Lambda_p} \leq \|T^*\|_{\Upsilon_{p^*}}$. □

Theorem 2.11 *An operator $S : F \to Y$ is positive p-majorizing if and only if S^* is positive p^*-summing. In this case, $\|S\|_{\Upsilon_p} = \|S^*\|_{\Lambda_{p^*}}$.*

Proof Assume that $S : F \to Y$ is positive p-majorizing with $\|S\|_{\Upsilon_p} \leq C$. Given any finite families $(y_j^*)_{j=1}^n$ in $(Y^*)_+$. For each $1 \leq j \leq n$, we choose v_j^{**} in $B_{F^{**}}$ with $\langle v_j^{**}, S^* y_j^*\rangle = \|S^* y_j^*\|$. Let $M = \text{span}\{v_j^{**} : 1 \leq j \leq n\}$ and $N = \text{span}\{S^* y_j^* : 1 \leq j \leq n\}$. Let $\epsilon > 0$. It follows from the principal of local reflexivity in Banach spaces that there exists an operator $R : M \to F$ such that

(i) $R|_{M\cap F} = I_{M\cap F}$;
(ii) $(1-\epsilon)\|v^{**}\| \leq \|Rv^{**}\| \leq (1+\epsilon)\|v^{**}\|$, $v^{**} \in M$;
(iii) $\langle v^*, Rv^{**}\rangle = \langle v^{**}, v^*\rangle$, $v^* \in N$, $v^{**} \in M$.

Let $v_j = Rv_j^{**} (j = 1, 2, \ldots, n)$. Then we have

$$\left(\sum_{j=1}^{n} \|S^* y_j^*\|^{p^*}\right)^{\frac{1}{p^*}} = \left(\sum_{j=1}^{n} |\langle v_j^{**}, S^* y_j^*\rangle|^{p^*}\right)^{\frac{1}{p^*}} = \left(\sum_{j=1}^{n} |\langle y_j^*, Sv_j\rangle|^{p^*}\right)^{\frac{1}{p^*}}$$
$$\leq C(1+\epsilon)\|(y_j^*)_{j=1}^n\|_{p^*}^w.$$

Letting $\epsilon \to 0$, we get

$$Z\left(\sum_{j=1}^{n} \|S^* y_j^*\|^{p^*}\right)^{\frac{1}{p^*}} \leq C\|(y_j^*)_{j=1}^n\|_{p^*}^w.$$

Hence S^* is positive p^*-summing and $\|S^*\|_{\Lambda_{p^*}} \leq \|S\|_{\Upsilon_p}$.

The converse follows from Theorem 2.10 and $\|S\|_{\Upsilon_p} \leq \|S^{**}\|_{\Upsilon_p} = \|S^*\|_{\Lambda_{p^*}}$.

□

The following two corollaries are immediate combinations of Theorems 2.10 and 2.11.

Corollary 2.12 [10] *An operator $T : X \to E$ is positive p-summing if and only if T^{**} is positive p-summing. In this case, $\|T\|_{\Lambda_p} = \|T^{**}\|_{\Lambda_p}$.*

Corollary 2.13 *An operator $S : E \to X$ is positive p-majorizing if and only if S^{**} is positive p-majorizing. In this case, $\|S\|_{\Upsilon_p} = \|S^{**}\|_{\Upsilon_p}$.*

At the rest of this section, we describe positive p-summing operators and positive p-majorizing operators in terms of suitable tensor products. Let us first recall the definition of reasonable cross norms.

Definition 2.14 [9] Let E, F be Banach spaces. A norm α on the algebraic tensor product $E \otimes F$ is called a *reasonable cross norm* if α satisfies the following conditions:

(a) $\alpha(u \otimes v) \leq \|u\|\|v\|$ for all $u \in E, v \in F$;
(b) $u^* \otimes v^* \in (E \otimes_\alpha F)^*$ and $\|u^* \otimes v^*\| \leq \|u^*\|\|v^*\|$ for all $u^* \in E^*, v^* \in F^*$.

Let E be a Banach space and let X be a Banach lattice. We define

$$|g|_p(z) := \inf \left\{ \|(u_i)_{i=1}^n\|_{p^*} \|(|x_i|)_{i=1}^n\|_p^w : z = \sum_{i=1}^n u_i \otimes x_i \right\}$$

for all $z \in E \otimes X$, and

$$|d|_p(z) := \inf \left\{ \|(|x_i|)_{i=1}^n\|_p^w \|(u_i)_{i=1}^n\|_{p^*} : z = \sum_{i=1}^n x_i \otimes u_i \right\}$$

for all $z \in X \otimes E$.

Arguments analogous to that of [7, Proposition 12.5] show that both $|g|_p$ and $|d|_p$ are reasonable cross norms. For $\varphi \in (E \otimes_{|g|_p} X^*)^*$, the associated operator $S_\varphi : E \to X^{**}$ defined by $\langle S_\varphi u, x^* \rangle = \langle \varphi, u \otimes x^* \rangle (u \in E, x^* \in X^*)$ has norm $\|S_\varphi\| \leq \|\varphi\|$. Hence $(E \otimes_{|g|_p} X^*)^*$ can be considered to be a linear subspace of $\mathcal{L}(E, X^{**})$. Similarly, the space $(X \otimes_{|d|_p} E^*)^*$ can also be considered to be a linear subspace of $\mathcal{L}(X, E^{**})$.

Theorem 2.15 $\Upsilon_p(E, X) = (E \otimes_{|g|_{p^*}} X^*)^* \cap \mathcal{L}(E, X)$ *holds isomorphically.*

Proof Let $S \in \Upsilon_p(E, X)$. We define $\varphi_S \in (E \otimes_{|g|_{p^*}} X^*)^*$ by $\varphi_S(u \otimes x^*) = \langle x^*, Su \rangle (u \in E, x^* \in X^*)$. Let $z = \sum_{i=1}^n u_i \otimes x_i^* \in E \otimes X^*$. By Lemma 2.6, we get

$$\begin{aligned} |\varphi_S(z)| &\leq \left(\sum_{i=1}^n \left| \left\langle x_i^*, S\left(\frac{u_i}{\|u_i\|} \right) \right\rangle \right|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^n \|u_i\|^p \right)^{\frac{1}{p}} \\ &\leq 2\|S\|_{\Upsilon_p} \|(|x_i^*|)_{i=1}^n\|_{p^*}^w \|(u_i)_{i=1}^n\|_p. \end{aligned}$$

Therefore, $|\varphi_S(z)| \leq 2\|S\|_{\Upsilon_p} |g|_{p^*}(z)$ and hence $\|\varphi_S\| \leq 2\|S\|_{\Upsilon_p}$. Clearly, $\varphi_S \in \mathcal{L}(E, X)$.

On the other hand, for $\varphi \in (E \otimes_{|g|_{p^*}} X^*)^* \cap \mathcal{L}(E, X)$, there exists an operator $S : E \to X$ such that $\langle \varphi, u \otimes x^* \rangle = \langle x^*, Su \rangle$ for all $u \in E, x^* \in X^*$. Given any $u_1, u_2, \ldots, u_n \in B_E$ and $x_1^*, x_2^*, \ldots, x_n^* \in (X^*)_+$. We choose $(\lambda_i)_{i=1}^n$ with $\sum_{i=1}^n |\lambda_i|^p = 1$ such that $(\sum_{i=1}^n |\langle x_i^*, Su_i \rangle|^{p^*})^{\frac{1}{p^*}} = \sum_{i=1}^n \lambda_i \langle x_i^*, Su_i \rangle$. Hence, we get

$$\left(\sum_{i=1}^n |\langle x_i^*, Su_i \rangle|^{p^*}\right)^{\frac{1}{p^*}} = \varphi\left(\sum_{i=1}^n \lambda_i u_i \otimes x_i^*\right)$$
$$\leq \|\varphi\| \|(\lambda_i)_{i=1}^n\|_p \|(x_i^*)_{i=1}^n\|_{p^*}^w = \|\varphi\| \|(x_i^*)_{i=1}^n\|_{p^*}^w.$$

This implies that S is positive p-majorizing and $\|S\|_{\Upsilon_p} \leq \|\varphi\|$.

□

A proof similar to that of Theorem 2.15 yields the following theorem.

Theorem 2.16 $\Lambda_p(X, E) = (X \otimes_{|d|_p} E^*)^* \cap \mathcal{L}(X, E)$ *holds isomorphically.*

3 Positive (p, q)-dominated operators

Definition 3.1 Let $1 \leq p, q \leq \infty$ and let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. We say that an operator T from a Banach lattice X to a Banach lattice Y is *positive* (p, q)-*dominated* if there exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^n |\langle y_i^*, Tx_i \rangle|^r\right)^{\frac{1}{r}} \leq C \|(x_i)_{i=1}^n\|_p^w \|(y_i^*)_{i=1}^n\|_q^w$$

for all finite families of $x_1, x_2, \ldots, x_n \in X_+$ and $y_1^*, y_2^*, \ldots, y_n^* \in (Y^*)_+$.

We put

$$\|T\|_{\Psi_{(p,q)}} := \inf C.$$

The class of all positive (p, q)-dominated operators from X to Y is denoted by $\Psi_{(p,q)}(X, Y)$.

In the sequel, we need the following simple, but useful result.

Theorem 3.2 *An operator $T : X \to Y$ is positive (p, q)-dominated if and only if there exists a constant $K > 0$ such that*

$$\left(\sum_{i=1}^n |\langle y_i^*, Tx_i \rangle|^r\right)^{\frac{1}{r}} \leq K \|(|x_i|)_{i=1}^n\|_p^w \|(|y_i^*|)_{i=1}^n\|_q^w$$

for all $x_1, x_2, \ldots, x_n \in X$, $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$ and all $n \in \mathbb{N}$.

Proof The sufficiency is trivial.

For the necessity, suppose that $T : X \to Y$ is positive (p,q)-dominated. Given any $x_1, x_2, \ldots, x_n \in X$ and $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$. Then one has

$$\begin{aligned}\left(\sum_{i=1}^n |\langle y_i^*, Tx_i\rangle|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{i=1}^n |\langle y_i^*, Tx_i^+\rangle|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n |\langle y_i^*, Tx_i^-\rangle|^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{i=1}^n |\langle (y_i^*)^+, Tx_i^+\rangle|^r\right)^{\frac{1}{r}} \\ &\quad + \left(\sum_{i=1}^n |\langle (y_i^*)^-, Tx_i^+\rangle|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n |\langle (y_i^*)^+, Tx_i^-\rangle|^r\right)^{\frac{1}{r}} \\ &\quad + \left(\sum_{i=1}^n |\langle (y_i^*)^-, Tx_i^-\rangle|^r\right)^{\frac{1}{r}} \\ &\leq 4\|T\|_{\Psi_{(p,q)}} \|(|x_i|)_{i=1}^n\|_p^w \|(|y_i^*|)_{i=1}^n\|_q^w.\end{aligned}$$

This completes the proof. □

The following theorem is a positive analogue of [18, Theorem 17.4.2]. Since its proof is similar to that of [18, Theorem 17.4.2], we make a sketch of the proof for the sake of completeness.

Theorem 3.3 *Let $1 \leq p, q < \infty$. An operator $T : X \to Y$ is positive (p,q)-dominated with constant C if and only if there exist a probability measure μ on $(B_{X^*})_+$ and a probability measure ν on $(B_{Y^{**}})_+$ such that*

$$|\langle y^*, Tx\rangle| \leq C\left[\int_{(B_{X^*})_+} \langle x^*, x\rangle^p d\mu(x^*)\right]^{\frac{1}{p}} \left[\int_{(B_{Y^{**}})_+} \langle y^{**}, y^*\rangle^q d\nu(y^{**})\right]^{\frac{1}{q}}$$

for all $x \in X_+$ and $y^* \in (Y^*)_+$.

Proof Since the sufficiency is evident, we only sketch the necessity.

We denote by $\mathcal{P}((B_{X^*})_+)$ (resp. $\mathcal{P}((B_{Y^{**}})_+)$) the family of all probability measures on $(B_{X^*})_+$ (resp. $(B_{Y^{**}})_+$). Then $\mathcal{P}((B_{X^*})_+) \times \mathcal{P}((B_{Y^{**}})_+)$ is a compact convex subset of the Hausdorff topological vector space $(C((B_{X^*})_+)^*, w^*) \times (C((B_{Y^{**}})_+)^*, w^*)$. For any finite families of $x_1, x_2, \ldots, x_n \in X_+$ and $y_1^*, y_2^*, \ldots, y_n^* \in (Y^*)_+$, we let

$$\begin{aligned}\Phi_{(x_1,x_2,\ldots,x_n;y_1^*,y_2^*,\ldots,y_n^*)}(\mu,\nu) = \sum_{j=1}^n \Bigg[\frac{1}{r}|\langle y_j^*, Tx_j\rangle|^r - \frac{C^r}{p}\int_{(B_{X^*})_+} \langle x^*, x_j\rangle^p d\mu(x^*) \\ - \frac{C^r}{q}\int_{(B_{Y^{**}})_+} \langle y^{**}, y_j^*\rangle^q d\nu(y^{**})\Bigg].\end{aligned}$$

for $(\mu, \nu) \in \mathcal{P}((B_{X^*})_+) \times \mathcal{P}((B_{Y^{**}})_+)$.

Clearly, each Φ is convex and continuous. Furthermore, the collection $\mathcal{F}$ of all such functions Φ is convex. Given $x_1, x_2, \dots, x_n \in X_+$ and $y_1^*, y_2^*, \dots, y_n^* \in (Y^*)_+$. Since $T : X \to Y$ is positive (p, q)-dominated with constant C, we get

$$\left(\sum_{i=1}^{n} |\langle y_i^*, Tx_i\rangle|^r\right)^{\frac{1}{r}} \leq C\|(x_i)_{i=1}^n\|_p^w \|(y_i^*)_{i=1}^n\|_q^w.$$

By (1.1), there exist $x_0^* \in (B_{X^*})_+$ and $y_0^{**} \in (B_{Y^{**}})_+$ so that $\|(x_i)_{i=1}^n\|_p^w = (\sum_{i=1}^n \langle x_0^*, x_i\rangle^p)^{\frac{1}{p}}$ and $\|(y_i^*)_{i=1}^n\|_q^w = (\sum_{i=1}^n \langle y_0^{**}, y_i^*\rangle^q)^{\frac{1}{q}}$. If $\delta_{x_0^*}$ and $\delta_{y_0^{**}}$ denotes the Dirac measure at x_0^* and y_0^{**}, respectively, then we have

$$\begin{aligned}
&\frac{1}{C^r}\sum_{i=1}^{n} |\langle y_i^*, Tx_i\rangle|^r \\
&\leq \left[\int_{(B_{X^*})_+} \sum_{i=1}^{n} \langle x^*, x_i\rangle^p d\delta_{x_0^*}(x^*)\right]^{\frac{r}{p}} \left[\int_{(B_{Y^{**}})_+} \sum_{i=1}^{n} \langle y^{**}, y_i^*\rangle^q d\delta_{y_0^{**}}(y^{**})\right]^{\frac{r}{q}} \\
&\leq \frac{r}{p}\int_{(B_{X^*})_+} \sum_{i=1}^{n} \langle x^*, x_i\rangle^p d\delta_{x_0^*}(x^*) \\
&\quad + \frac{r}{q}\int_{(B_{Y^{**}})_+} \sum_{i=1}^{n} \langle y^{**}, y_i^*\rangle^q d\delta_{y_0^{**}}(y^{**}).
\end{aligned}$$

Hence, $\Phi_{(x_1,x_2,\dots,x_n;y_1^*,y_2^*,\dots,y_n^*)}(\delta_{x_0^*}, \delta_{y_0^{**}}) \leq 0$. It follows from Ky Fan's Lemma that there exist $\mu \in \mathcal{P}((B_{X^*})_+)$ and $\nu \in \mathcal{P}((B_{Y^{**}})_+)$ so that $\Phi_{(x_1,x_2,\dots,x_n;y_1^*,y_2^*,\dots,y_n^*)}(\mu, \nu) \leq 0$ for any finite sequences $x_1, x_2, \dots, x_n \in X_+$ and $y_1^*, y_2^*, \dots, y_n^* \in (Y^*)_+$. In particular,

$$\frac{1}{r}|\langle y^*, Tx\rangle|^r \leq C^r\left[\frac{1}{p}\int_{(B_{X^*})_+} \langle x^*, x\rangle^p d\mu(x^*) + \frac{1}{q}\int_{(B_{Y^{**}})_+} \langle y^{**}, y^*\rangle^q d\nu(y^{**})\right] \tag{3.1}$$

for all $x \in X_+$, $y^* \in (Y^*)_+$.

Finally, applying $[\int_{(B_{X^*})_+} \langle x^*, x\rangle^p d\mu(x^*)]^{-\frac{1}{p}} x$ and $[\int_{(B_{Y^{**}})_+} \langle y^{**}, y^*\rangle^q d\nu(y^{**})]^{-\frac{1}{q}} y^*$ to (3.1), we obtain

$$|\langle y^*, Tx\rangle| \leq C\left[\int_{(B_{X^*})_+} \langle x^*, x\rangle^p d\mu(x^*)\right]^{\frac{1}{p}} \left[\int_{(B_{Y^{**}})_+} \langle y^{**}, y^*\rangle^q d\nu(y^{**})\right]^{\frac{1}{q}}$$

for all $x \in X_+$ and $y^* \in (Y^*)_+$.

□

The following is an immediate consequence of Theorem 3.3. We omit the straightforward proof.

Corollary 3.4 *Let $1 \leq p, q < \infty$. An operator $T : X \to Y$ is positive (p, q)-dominated if and only if there exist a constant $K > 0$, a probability measure μ on $(B_{X^*})_+$ and a probability measure ν on $(B_{Y^{**}})_+$ such that*

$$|\langle y^*, Tx\rangle| \leq K\left[\int_{(B_{X^*})_+} \langle x^*, |x|\rangle^p d\mu(x^*)\right]^{\frac{1}{p}} \left[\int_{(B_{Y^{**}})_+} \langle y^{**}, |y^*|\rangle^q d\nu(y^{**})\right]^{\frac{1}{q}}$$

for all $x \in X$ and $y^* \in Y^*$.

To prove the main result of this section, we need a function space introduced by Achour and Belacel [1].

Let X be a Banach lattice and let μ be a regular Borel probability measure on $(B_{X^*})_+$. We denote by i_X the canonical embedding $X \to C((B_{X^*})_+)$ given by $\langle i_X(x), x^*\rangle = \langle x^*, x\rangle (x \in X, x^* \in (B_{X^*})_+)$. It is easy to see that

$$\frac{1}{2}\|x\| \leq \|i_X(x)\| \leq \|x\|, x \in X.$$

Let us define a seminorm $|||\cdot|||_p$ on vector space $i_X(X)$ by

$$|||f_x|||_p = \left[\int_{(B_{X^*})_+} \langle x^*, |x|\rangle^p d\mu(x^*)\right]^{\frac{1}{p}}, \quad f_x = i_X(x) \in i_X(X).$$

We set

$$R_p = \{f_x \in i_X(X) : |||f_x|||_p = 0\}.$$

Then $i_X(X)/R_p$ becomes a normed space under the norm

$$\|[f_x]\|_p = |||f_x|||_p, \quad f_x \in i_X(X).$$

Let $L_0^p(\mu)$ be the completion of the normed space $(i_X(X)/R_p, |||\cdot|||_p)$.

We define

$$J_{p,0} : i_X(X) \to L_0^p(\mu), f_x \mapsto [f_x].$$

Then $J_{p,0}$ is linear and bounded. Indeed, for $x \in X$, we get

$$\begin{aligned}\left[\int_{(B_{X^*})_+} \langle x^*, |x|\rangle^p d\mu(x^*)\right]^{\frac{1}{p}} &\le \sup_{x^*\in(B_{X^*})_+} \langle x^*, |x|\rangle \\ &= \sup_{x^*\in(B_{X^*})_+} \sup\{|\langle y^*, x\rangle| : |y^*| \le x^*, y^* \in X^*\} \\ &= \sup\{|\langle z^*, x\rangle| : z^* \in B_{X^*}\} \\ &\le 2 \sup_{x^*\in(B_{X^*})_+} |\langle x^*, x\rangle| \\ &= 2\|i_X(x)\|\end{aligned}$$

Furthermore, Achour and Belacel [1, Lemma3.5] proved that the operator $J_{p,0}i_X : X \to i_X(X) \to L_0^p(\mu)$ is positive p-summing with constant 1.

We also need the Pietsch domination theorem for positive p-summing operators due to O. I. Zhukova [22].

Theorem 3.5 [22] *An operator $T : X \to E$ is positive p-summing with $\|T\|_{\Lambda_p} \le C$ if and only if there exist a probability measure μ on $(B_{X^*})_+$ such that*

$$\|Tx\| \le C\left[\int_{(B_{X^*})_+} \langle x^*, x\rangle^p d\mu(x^*)\right]^{\frac{1}{p}}$$

for all $x \in X_+$.

Achour and Belacel [1] made a slight improvement of Theorem 3.5.

Theorem 3.6 [1] *An operator $T : X \to E$ is positive p-summing if and only if there exist a constant $K > 0$ and a probability measure μ on $(B_{X^*})_+$ such that*

$$\|Tx\| \le K\left[\int_{(B_{X^*})_+} \langle x^*, |x|\rangle^p d\mu(x^*)\right]^{\frac{1}{p}}$$

for all $x \in X$.

We are now in a position to prove the main result of this section.

Theorem 3.7 $\Psi_{(p,q)} = \Upsilon_{q^*} \circ \Lambda_p$.

Proof Suppose that $T : X \to E$ is positive p-summing and $S : E \to Y$ is positive q^*-majorizing. Given any $x_1, x_2, \dots, x_n$ in X_+ and $y_1^*, y_2^*, \dots, y_n^*$ in $(Y^*)_+$. By Theorem 2.11, we have

$$\begin{aligned}\left(\sum_{i=1}^n |\langle y_i^*, STx_i\rangle|^r\right)^{\frac{1}{r}} &\le \left(\sum_{i=1}^n \|Tx_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|S^*y_i^*\|^q\right)^{\frac{1}{q}} \\ &\le \|T\|_{\Lambda_p}\|(x_i)_{i=1}^n\|_p^w \|S^*\|_{\Lambda_q}\|(y_i^*)_{i=1}^n\|_q^w.\end{aligned}$$

This implies that ST is positive (p, q)-dominated. Hence $\Upsilon_{q^*} \circ \Lambda_p \subseteq \Psi_{(p,q)}$.

Conversely, assume that $T : X \to Y$ is positive (p, q)-dominated. It follows from Corollary 3.4 that there exist a constant $K > 0$, a probability measure μ on $(B_{X^*})_+$ and a probability measure ν on $(B_{Y^{**}})_+$ such that

$$|\langle y^*, Tx\rangle| \leq K\left[\int_{(B_{X^*})_+} \langle x^*, |x|\rangle^p d\mu(x^*)\right]^{\frac{1}{p}} \left[\int_{(B_{Y^{**}})_+} \langle y^{**}, |y^*|\rangle^q d\nu(y^{**})\right]^{\frac{1}{q}} \tag{3.2}$$

for all $x \in X$ and $y^* \in Y^*$.

We define an operator

$$R : i_X(X)/R_p \to Y, [f_x] \mapsto Tx,\ f_x = i_X(x), x \in X.$$

By (3.2), we see that R is well-defined and bounded with $\|R\| \leq K$. We extend R to an operator defined on $L_0^p(\mu)$, which is still denoted by R. It is clear that $T = R \circ (J_{p,0} \circ i_X)$. Since the operator $J_{p,0} \circ i_X$ is positive p-summing, it remains to prove that R is positive q^*-majorizing.

Let $y^* \in Y^*$. By (3.2), we obtain

$$\begin{aligned}\|R^* y^*\| &= \sup\{|\langle R^* y^*, [f_x]\rangle| : [f_x] \in B_{i_X(X)/R_p}, x \in X\} \\ &= \sup\{|\langle y^*, Tx\rangle| : \left[\int_{(B_{X^*})_+} \langle x^*, |x|\rangle^p d\mu(x^*)\right]^{\frac{1}{p}} \leq 1, x \in X\} \\ &\leq K\left[\int_{(B_{Y^{**}})_+} \langle y^{**}, |y^*|\rangle^q d\nu(y^{**})\right]^{\frac{1}{q}}\end{aligned}$$

Theorem 3.6 ensures that R^* is positive q-summing. Again by Theorem 2.11, R is positive q^*-majorizing. This completes the proof.

□

Theorem 3.7 teamed with Theorems 2.10 and 2.11 results in

Corollary 3.8 *Let $T : X \to Y$ be an operator. The following are equivalent:*

(1) *T is positive (p, q)-dominated;*
(2) *T^* is positive (q, p)-dominated;*
(3) *T^{**} is positive (p, q)-dominated.*

Proof (1) $\Rightarrow$ (2). Suppose that T is positive (p, q)-dominated. By Theorem 3.7, there exist a Banach space E, a positive p-summing operator $R : X \to E$ and a positive q^*-majorizing operator $S : E \to Y$ such that $T = SR$. An appeal to Theorem 2.10 shows that R^* is positive p^*-majorizing. By virtue of Theorem 2.11, S^* is positive q-summing. Again by Theorem 3.7, $T^* = R^* S^*$ is positive (q, p)-dominated.

(2) $\Rightarrow$ (3) clearly follows from (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1). Assume that T^{**} is positive (p,q)-dominated. It is obvious that $J_Y T = T^{**} J_X$ is positive (p,q)-dominated. Then there exists a constant $C > 0$ such that

$$\left(\sum_{j=1}^{n} |\langle y_j^*, Tx_j\rangle|^r\right)^{\frac{1}{r}} = \left(\sum_{j=1}^{n} |\langle J_{Y^*} y_j^*, J_Y T x_j\rangle|^r\right)^{\frac{1}{r}} \leq C\|(x_j)_{j=1}^n\|_p^w \|(y_j^*)_{j=1}^n\|_q^w.$$

for any finitely many $x_1, x_2, \ldots, x_n \in X_+$ and $y_1^*, y_2^*, \ldots, y_n^* \in (Y^*)_+$.

Hence T is positive (p,q)-dominated and the proof is completed. □

Next we describe the positive (p,q)-dominated operators in terms of a tensor product, for which we introduce the following reasonable cross norm.

Suppose that $\frac{1}{p} + \frac{1}{q} \geq 1$. Let $r \in [1, +\infty]$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. We set

$$|\alpha|_{p,q}(z) = \inf\left\{\|(\lambda_i)_{i=1}^n\|_r \|(|x_i|)_{i=1}^n\|_{q^*}^w \|(|y_i|)_{i=1}^n\|_{p^*}^w : z = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i\right\}$$

for $z \in X \otimes Y$.

An argument similar to that of [7, Proposition 12.5] shows that $|\alpha|_{p,q}$ is a reasonable cross norm on $X \otimes Y$. For $\varphi \in (X \otimes_{|\alpha|_{p,q}} Y^*)^*$, we define an operator $S_\varphi : X \to Y^{**}$ by $\langle S_\varphi x, y^*\rangle = \langle \varphi, x \otimes y^*\rangle$ for all $x \in X$, $y^* \in Y^*$. Clearly, $\|S_\varphi\| \leq \|\varphi\|$. Thus the space $(X \otimes_{|\alpha|_{p,q}} Y^*)^*$ can be considered to be a linear subspace of $\mathcal{L}(X, Y^{**})$.

Theorem 3.9 $\Psi_{(p,q)}(X,Y) = (X \otimes_{|\alpha|_{q^*,p^*}} Y^*)^* \cap \mathcal{L}(X,Y)$ *holds isomorphically.*

Proof Let us define the operator

$$V : \Psi_{(p,q)}(X,Y) \to (X \otimes_{|\alpha|_{q^*,p^*}} Y^*)^* \cap \mathcal{L}(X,Y), \quad T \mapsto \varphi_T,$$

where $\varphi_T(z) = \sum_{i=1}^n \lambda_i \langle y_i^*, Tx_i\rangle$, $z = \sum_{i=1}^n \lambda_i x_i \otimes y_i^* \in X \otimes Y^*$. By Theorem 3.2, we get

$$\begin{aligned}|\varphi_T(z)| &\leq \sum_{i=1}^{n} |\lambda_i||\langle y_i^*, Tx_i\rangle| \\ &\leq \left(\sum_{i=1}^{n} |\lambda_i|^{r^*}\right)^{\frac{1}{r^*}} \left(\sum_{i=1}^{n} |\langle y_i^*, Tx_i\rangle|^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{i=1}^{n} |\lambda_i|^{r^*}\right)^{\frac{1}{r^*}} 4\|T\|_{\Psi_{(p,q)}} \|(|x_i|)_{i=1}^n\|_p^w \|(|y_i^*|)_{i=1}^n\|_q^w.\end{aligned}$$

This means that $|\varphi_T(z)| \leq 4\|T\|_{\Psi_{(p,q)}} |\alpha|_{q^*,p^*}(z)$. Hence $\|\varphi_T\| \leq 4\|T\|_{\Psi_{(p,q)}}$.

For the converse, let $\varphi \in (X \otimes_{|\alpha|_{q^*,p^*}} Y^*)^* \cap \mathcal{L}(X, Y)$. Then there exists an operator $T : X \to Y$ such that $\langle \varphi, x \otimes y^* \rangle = \langle y^*, Tx \rangle$ for all $x \in X$, $y^* \in Y^*$. Thus, we get $\varphi_T = \varphi$. It remains to prove that T is positive (p, q)-dominated and $\|T\|_{\Psi_{(p,q)}} \leq \|\varphi\|$. Indeed, given any $x_1, x_2, \ldots, x_n \in X_+$ and $y_1^*, y_2^*, \ldots, y_n^* \in (Y^*)_+$. We choose $(\lambda_i)_{i=1}^n \in B_{l_{r^*}^n}$ such that $(\sum_{i=1}^n |\langle y_i^*, Tx_i \rangle|^r)^{\frac{1}{r}} = \sum_{i=1}^n \lambda_i \langle y_i^*, Tx_i \rangle$. Thus, we get

$$\left(\sum_{i=1}^n |\langle y_i^*, Tx_i \rangle|^r \right)^{\frac{1}{r}} = \left\langle \varphi, \sum_{i=1}^n \lambda_i x_i \otimes y_i^* \right\rangle \leq \|\varphi\| \|(x_i)_{i=1}^n\|_p^w \|(y_i)_{i=1}^n\|_q^w .$$

Therefore T is positive (p, q)-dominated and $\|T\|_{\Psi_{(p,q)}} \leq \|\varphi\|$.

□

4 Disjoint *p*-summing operators

Definition 4.1 We say that an operator $T : X \to E$ is *disjoint p-summing* if there exists a constant $C > 0$ such that for any choice of finitely many pairwise disjoint elements $x_1, x_2, \ldots, x_n$ in X, we have

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq C \|(x_i)_{i=1}^n\|_p^w . \tag{4.1}$$

We denote by $\Gamma_p(X, E)$ the space of all disjoint p-summing operators from X to E. A standard argument shows that $\Gamma_p(X, E)$ becomes a Banach space with the norm $\|\cdot\|_{\Gamma_p}$ given by the infimum of the constants satisfying (4.1). For $p = \infty$, $\Gamma_\infty(X, E) = \mathcal{L}(X, E)$ and $\|T\|_{\Gamma_\infty} = \|T\|$. Using the same proof as that of [8, Theorem 2.8], we easily get the inclusion result of disjoint p-summing operators: $\Gamma_p \subseteq \Gamma_q$ if $p < q$.

The following characterization of disjoint p-summing operators is straightforward.

Proposition 4.2 *An operator $T : X \to E$ is disjoint p-summing if and only if there exists a constant $K > 0$ such that for all pairwise disjoint positive elements $x_1, x_2, \ldots, x_n$ in X, we have*

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq K \left\| (x_i)_{i=1}^n \right\|_p^w .$$

Proof It suffices to prove the sufficient part. Given any pairwise disjoint elements $x_1, x_2, \ldots, x_n$ in X. Then $x_1^+, x_2^+, \ldots, x_n^+$ are pairwise disjoint and hence, we get

$$\left(\sum_{i=1}^n \|Tx_i^+\|^p \right)^{\frac{1}{p}} \leq K \|(x_i^+)_{i=1}^n\|_p^w .$$

Similarly, we get

$$\left(\sum_{i=1}^{n}\|Tx_i^-\|^p\right)^{\frac{1}{p}} \leq K\left\|(x_i^-)_{i=1}^n\right\|_p^w.$$

Thus

$$\begin{aligned}\left(\sum_{i=1}^{n}\|Tx_i\|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{n}\|Tx_i^+\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}\|Tx_i^-\|^p\right)^{\frac{1}{p}} \\ &\leq K(\|(x_i^+)_{i=1}^n\|_p^w + \|(x_i^-)_{i=1}^n\|_p^w) \\ &= K\left(\sup_{a\in(B_{l_{p^*}^n})_+}\left\|\sum_{i=1}^{n}a_ix_i^+\right\| + \sup_{a\in(B_{l_{p^*}^n})_+}\left\|\sum_{i=1}^{n}a_ix_i^-\right\|\right) \qquad (4.2)\end{aligned}$$

Since $x_1, x_2, \ldots, x_n$ are pairwise disjoint, we have

$$\left|\sum_{i=1}^{n}a_ix_i\right| = \sum_{i=1}^{n}a_i|x_i| = \sum_{i=1}^{n}a_ix_i^+ + \sum_{i=1}^{n}a_ix_i^-, \quad \forall a = (a_i)_{i=1}^n \in (B_{l_{p^*}^n})_+.$$

This implies

$$\sup_{a\in\left(B_{l_{p^*}^n}\right)_+}\left\|\sum_{i=1}^{n}a_ix_i^+\right\| \leq \sup_{a\in(B_{l_{p^*}^n})_+}\left\|\sum_{i=1}^{n}a_ix_i\right\| \qquad (4.3)$$

and

$$\sup_{a\in\left(B_{l_{p^*}^n}\right)_+}\left\|\sum_{i=1}^{n}a_ix_i^-\right\| \leq \sup_{a\in\left(B_{l_{p^*}^n}\right)_+}\left\|\sum_{i=1}^{n}a_ix_i\right\|. \qquad (4.4)$$

Combining inequalities (4.2), (4.3) and (4.4), we get

$$\left(\sum_{i=1}^{n}\|Tx_i\|^p\right)^{\frac{1}{p}} \leq 2K\sup_{a\in B_{l_{p^*}^n}}\left\|\sum_{i=1}^{n}a_ix_i\right\| = 2K\|(x_i)_{i=1}^n\|_p^w.$$

The conclusion follows. □

An easy consequence of Proposition 4.2 is the following relationship between positive p-summing operators and disjoint p-summing operators.

Corollary 4.3 $\Lambda_p \subseteq \Gamma_p$

Remark 4.4 We do not know whether the above inclusion is strict in general. But, when we look at operators on $L_p(\mu)$, we at least have the following partial result that is inspired by [4, Theorem 1].

Theorem 4.5 *Let* (Ω, Σ, μ) *be a finite measure space and* $1 \le p < \infty$. *Then*

$$\Lambda_1(L_{p^*}(\mu), E) = \Gamma_s(L_{p^*}(\mu), E)$$

for every Banach space E *and every* $s \in [1, p]$.

Proof By Corollary 4.3, it suffices to prove that $\Gamma_p(L_{p^*}(\mu), E) \subseteq \Lambda_1(L_{p^*}(\mu), E)$ if $1 < p < \infty$.

Let $T \in \Gamma_p(L_{p^*}(\mu), E)$ and define a vector measure $G : \Sigma \to E$ by $G(A) = T(\chi_A)$ for $A \in \Sigma$. Lebesgue's dominated convergence theorem ensures that G is countably additive. Let $A \in \Sigma$ and let $\pi = (A_i)_{i=1}^n$ be a partition of A. Then we have

$$\begin{aligned}\sum_{i=1}^n \|G(A_i)\| &= \sum_{i=1}^n \|T(\chi_{A_i})\| \\ &= \sum_{i=1}^n \|T(\mu(A_i)^{-\frac{1}{p^*}} \chi_{A_i})\| \mu(A_i)^{\frac{1}{p^*}} \\ &\le \left(\sum_{i=1}^n \|T(\mu(A_i)^{-\frac{1}{p^*}} \chi_{A_i})\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \mu(A_i)\right)^{\frac{1}{p^*}} \\ &\le \|T\|_{\Gamma_p} \|(\mu(A_i)^{-\frac{1}{p^*}} \chi_{A_i})_{i=1}^n\|_p^w \mu(A)^{\frac{1}{p^*}} \\ &= \|T\|_{\Gamma_p} \mu(A)^{\frac{1}{p^*}}\end{aligned}$$

Hence $|G|(A) \le \|T\|_{\Gamma_p} \mu(A)^{\frac{1}{p^*}}$ for all $A \in \Sigma$. This implies that $|G|$ is a finite measure and is absolutely continuous with respect to μ. The Radon–Nikodým theorem yields a Σ-measurable function $g : \Omega \to [0, +\infty)$ such that $|G|(A) = \int_A g d\mu$ for all $A \in \Sigma$.

Claim 1 $g \in L_p(\mu)$.

Indeed, for each $n \in \mathbb{N}$, we set $B_n = \{\omega \in \Omega : g(\omega) \le n\}$. By the monotone convergence theorem, we get

$$\lim_{n \to \infty} \int_\Omega (g\chi_{B_n})^p d\mu = \int_\Omega g^p d\mu. \tag{4.5}$$

For each $n \in \mathbb{N}$, we set $h_n = (g\chi_{B_n})^{p-1}$. Then one has

$$\int_\Omega h_n(g\chi_{B_n}) d\mu = \int_\Omega (g\chi_{B_n})^p d\mu \tag{4.6}$$

and

$$\int_\Omega h_n^{p^*} d\mu = \int_\Omega (g\chi_{B_n})^p d\mu. \tag{4.7}$$

Let $\epsilon > 0$. We choose a simple Σ-measurable function $f_n = \sum_{i=1}^{m_n} a_i^{(n)} \chi_{A_i^{(n)}}$, where $(A_i^{(n)})_{i=1}^{m_n}$ are pairwise disjoint measurable subsets, such that $\|h_n - f_n\|_{p^*} < \epsilon$. By (4.6), we get

$$\begin{aligned}
\int_\Omega (g\chi_{B_n})^p d\mu &= \int_\Omega h_n (g\chi_{B_n}) d\mu \\
&\leq \int_\Omega |h_n - f_n| (g\chi_{B_n}) d\mu + \int_\Omega |f_n| (g\chi_{B_n}) d\mu \\
&\leq \|h_n - f_n\|_{p^*} \|g\chi_{B_n}\|_p + \sum_{i=1}^{m_n} |a_i^{(n)}| \int_\Omega \chi_{A_i^{(n)}} g\chi_{B_n} d\mu \\
&\leq \epsilon \|g\chi_{B_n}\|_p + \sum_{i=1}^{m_n} |a_i^{(n)}| |G|(A_i^{(n)} \cap B_n) \\
&\leq \epsilon \|g\chi_{B_n}\|_p + \sum_{i=1}^{m_n} |a_i^{(n)}| |G|(A_i^{(n)})
\end{aligned} \tag{4.8}$$

Next we'll prove that

$$\sum_{i=1}^{m_n} |a_i^{(n)}| |G|(A_i^{(n)}) \leq (\epsilon + \|h_n\|_{p^*}) \|T\|_{\Gamma_p}.$$

First note that

$$\|f_n\|_{p^*} = \left(\sum_{i=1}^{m_n} \left| a_i^{(n)} \right|^{p^*} \mu\left(A_i^{(n)}\right) \right)^{\frac{1}{p^*}} \leq \epsilon + \|h_n\|_{p^*}.$$

Let $\delta > 0$ be arbitrary. For each $i = 1, 2, \dots, m_n$, we choose a partition $(C_j^{(i)})_{j=1}^{n_i}$ of $A_i^{(n)}$ such that

$$\sum_{j=1}^{n_i} \left\| G\left(C_j^{(i)}\right) \right\| > |G|\left(A_i^{(n)}\right) - \delta. \tag{4.9}$$

Since T is disjoint p-summing, we get

$$\sum_{i=1}^{m_n} |a_i^{(n)}| \sum_{j=1}^{n_i} \|G(C_j^{(i)})\| = \sum_{i=1}^{m_n} \sum_{j=1}^{n_i} |a_i^{(n)}| \mu(C_j^{(i)})^{\frac{1}{p^*}} \|T(\mu(C_j^{(i)})^{-\frac{1}{p^*}} \chi_{C_j^{(i)}})\|$$

$$\leq \left(\sum_{i=1}^{m_n} \sum_{j=1}^{n_i} |a_i^{(n)}|^{p^*} \mu(C_j^{(i)}) \right)^{\frac{1}{p^*}}$$

$$\left(\sum_{i=1}^{m_n} \sum_{j=1}^{n_i} \| T(\mu(C_j^{(i)})^{-\frac{1}{p^*}} \chi_{C_j^{(i)}}) \|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{m_n} |a_i^{(n)}|^{p^*} \mu(A_i^{(n)}) \right)^{\frac{1}{p^*}}$$

$$\|T\|_{\Gamma_p} \| (\mu(C_j^{(i)})^{-\frac{1}{p^*}} \chi_{C_j^{(i)}})_{i=1, j=1}^{m_n, n_i} \|_p^w$$

$$\leq (\epsilon + \|h_n\|_{p^*}) \|T\|_{\Gamma_p} \tag{4.10}$$

Combining (4.9) with (4.10), we get

$$\sum_{i=1}^{m_n} \left| a_i^{(n)} \right| |G| \left(A_i^{(n)} \right) \leq \delta \sum_{i=1}^{m_n} \left| a_i^{(n)} \right| + (\epsilon + \|h_n\|_{p^*}) \|T\|_{\Gamma_p}.$$

Letting $\delta \to 0$, we get

$$\sum_{i=1}^{m_n} |a_i^{(n)}| |G| (A_i^{(n)}) \leq (\epsilon + \|h_n\|_{p^*}) \|T\|_{\Gamma_p}. \tag{4.11}$$

By (4.8) and (4.11), we have

$$\int_\Omega (g \chi_{B_n})^p d\mu \leq \epsilon \|g \chi_{B_n}\|_p + (\epsilon + \|h_n\|_{p^*}) \|T\|_{\Gamma_p}.$$

Again letting $\epsilon \to 0$, we get

$$\int_\Omega (g \chi_{B_n})^p d\mu \leq \|h_n\|_{p^*} \|T\|_{\Gamma_p}.$$

By (4.7), we get

$$\|g \chi_{B_n}\|_p \leq \|T\|_{\Gamma_p}.$$

By (4.5), we conclude that $g \in L_p(\mu)$ with $\|g\|_p \leq \|T\|_{\Gamma_p}$.

Claim 2 $\|Tf\| \leq \int_\Omega |f| g d\mu$ for all $f \in L_{p^*}(\mu)$.

Let $f \in L_{p^*}(\mu)$. We may assume that $f \geq 0$. Then, for every $\epsilon > 0$, there exists a non-negative simple Σ-measurable $h = \sum_{j=1}^{n} c_j \chi_{D_j}$, where $(D_j)_{j=1}^n$ are pairwise disjoint, such that $\|f-h\|_{p^*}<\epsilon$. By Claim 1, we get

$$\begin{aligned}
\|Tf\| &\leq \|Tf-Th\|+\|Th\| \\
&\leq \epsilon\|T\|+\sum_{j=1}^{n} c_j\|G(D_j)\| \\
&\leq \epsilon\|T\|+\sum_{j=1}^{n} c_j\int_{D_j} g d\mu \\
&= \epsilon\|T\|+\int_{\Omega} ghd\mu \\
&= \epsilon\|T\|+\int_{\Omega} gfd\mu+\int_{\Omega} g(h-f)d\mu \\
&\leq \epsilon\|T\|+\int_{\Omega} gfd\mu+\|g\|_p\|h-f\|_{p^*} \\
&\leq \epsilon\|T\|+\int_{\Omega} gfd\mu+\epsilon\|g\|_p
\end{aligned}$$

Letting $\epsilon\to 0$, we conclude Claim 2.

Finally, we'll prove that T is positive 1-summing. Given any positive elements $f_1, f_2, \ldots, f_n$ in $L_{p^*}(\mu)$. By Claim 2, we get

$$\begin{aligned}
\sum_{i=1}^{n}\|Tf_i\| &\leq \int_{\Omega}\left(\sum_{i=}^{n} f_i\right) g d\mu \\
&\leq \left\|\sum_{i=}^{n} f_i\|_{p^*}\|g\right\|_p \\
&= \|(f_i)_{i=1}^n\|_1^w\|g\|_p \\
&\leq \|T\|_{\Gamma_p}\|(f_i)_{i=1}^n\|_1^w.
\end{aligned}$$

This completes the proof. □

Combining [4, Theorem 2] with Theorem 4.5, we derive the following corollary.

Corollary 4.6 *Let (Ω,Σ,μ) be a finite measure space and $1<p\leq\infty$. Then a Banach space E has the Radon–Nikodým property with respect to (Ω,Σ,μ) if and only if $\Gamma_p(L_{p^*}(\mu),E)=L_p(\mu,E)$.*

To characterize disjoint p-summing operators, we recall [17] that an operator $T: X\to Y$ is called *disjointness preserving* if $Tx_1\perp Tx_2$ for all $x_1,x_2\in X$ satisfying $x_1\perp x_2$. [17, Theorem 3.15] states that every disjointness preserving operator is regular.

Corollary 4.7 *Let $1\leq p<\infty$ and $C>0$. The following statements about an operator $T: X\to E$ are equivalent:*

(1) *T is disjoint p-summing with $\|T\|_{\Gamma_p} \le C$;*
(2) *$\|TR\|_{\Lambda_1} \le C\|R\|$ for each $n \in \mathbb{N}$ and each disjointness preserving operator $R : l^n_{p^*} \to X (R : c^n_0 \to X$ for $p = 1)$.*

Proof (1) $\Rightarrow$ (2). Suppose that T is disjoint p-summing with $\|T\|_{\Gamma_p} \le C$. Let $R : l^n_{p^*} \to X$ be disjointness preserving. It follows immediately that $TR : l^n_{p^*} \to E$ is disjoint p-summing. By Theorem 4.5, we get $\|TR\|_{\Lambda_1} \le \|TR\|_{\Gamma_p} \le C\|R\|$.

(2) $\Rightarrow$ (1). Given any choice of finitely many pairwise disjoint elements $x_1, x_2, \ldots, x_n$ in X. Let us define an operator $R : l^n_{p^*} \to X$ by $Re_i = x_i (i = 1, 2, \ldots, n)$. Let $(a_i)^n_{i=1}, (b_i)^n_{i=1} \in l^n_{p^*}$ be disjoint. This means that $a_i b_i = 0$ for all $i = 1, 2, \ldots, n$. Then we have

$$\left|\sum_{i=1}^n a_i x_i\right| \wedge \left|\sum_{i=1}^n b_i x_i\right| \le \left(\sum_{i=1}^n |a_i x_i|\right) \wedge \left(\sum_{i=1}^n |b_i x_i|\right) \le \sum_{i=1}^n \sum_{j=1}^n |a_i||b_j||x_i| \wedge |x_j| = 0.$$

Hence $(\sum_{i=1}^n a_i x_i)^n_{i=1} \perp (\sum_{i=1}^n b_i x_i)^n_{i=1}$. This implies that R is disjointness preserving. By (2), we get TR is positive 1-summing and $\|TR\|_{\Lambda_1} \le C\|R\| = C\|(x_i)^n_{i=1}\|^w_p$. Thus TR is positive p-summing and $\|TR\|_{\Lambda_p} \le \|TR\|_{\Lambda_1} \le C\|(x_i)^n_{i=1}\|^w_p$. Therefore we get

$$\left(\sum_{i=1}^n \|Tx_i\|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \|TRe_i\|^p\right)^{\frac{1}{p}} \le \|TR\|_{\Lambda_p} \|(e_i)^n_{i=1}\|^w_p \le C\|(x_i)^n_{i=1}\|^w_p.$$

□

We end this section by discovering the relationship between disjoint p-summing operator T and its second dual T^{**}.

Theorem 4.8 *An operator $T : X \to E$ is disjoint p-summing if and only if so is T^{**}. In this case, $\|T\|_{\Gamma_p} = \|T^{**}\|_{\Gamma_p}$.*

Proof Suppose that T is disjoint p-summing. Given any $y_1^*, y_2^*, \ldots, y_n^*$ in E^* and any pairwise disjoint elements $x_1^{**}, x_2^{**}, \ldots, x_n^{**}$ in X^{**}. Let M be the subspace generated by $(x_1^{**})^+, (x_2^{**})^+, \ldots, (x_n^{**})^+, (x_1^{**})^-, (x_2^{**})^-, \ldots, (x_n^{**})^-$. Then M is a $2n$-dimensional sublattice of X^{**}. We set $N = \text{span}\{T^* y_j^* : j = 1, 2, \ldots, n\}$. Let $\epsilon > 0$. Theorem 2.8 produces a lattice isomorphism R from M into X such that $\|R\|, \|R^{-1}\| \le 1 + \epsilon$ and

$$|\langle x^{**}, x^* \rangle - \langle x^*, Rx^{**} \rangle| \le \epsilon \|x^{**}\| \|x^*\|, \forall x^{**} \in M, \forall x^* \in N.$$

Let $x_j = Rx_j^{**} (j = 1, 2, \ldots, n)$. Then $x_1, x_2, \ldots, x_n$ are pairwise disjoint and hence

$$\left(\sum_{j=1}^{n}\left|\left\langle T^{**}x_j^{**}, y_j^*\right\rangle\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n}\left|\left\langle T^{**}x_j^{**}, y_j^*\right\rangle - \left\langle T^*y_j^*, Rx_j^{**}\right\rangle\right|^p\right)^{\frac{1}{p}}$$

$$+\left(\sum_{j=1}^{n}|\langle y_j^*, Tx_j\rangle|^p\right)^{\frac{1}{p}}$$

$$\leq \epsilon\left(\sum_{j=1}^{n}\|x_j^{**}\|^p\|T^*y_j^*\|^p\right)^{\frac{1}{p}}$$

$$+\|T\|_{\Gamma_p}\|(x_j)_{j=1}^n\|_p^w \sup_{1\leq j\leq n}\|y_j^*\|$$

$$\leq \epsilon\left(\sum_{j=1}^{n}\|x_j^{**}\|^p\|T^*y_j^*\|^p\right)^{\frac{1}{p}}$$

$$+\|T\|_{\Gamma_p}(1+\epsilon)\|(x_j^{**})_{j=1}^n\|_p^w \sup_{1\leq j\leq n}\|y_j^*\|$$

Letting $\epsilon \to 0$, we get

$$\left(\sum_{j=1}^{n}\left|\left\langle T^{**}x_j^{**}, y_j^*\right\rangle\right|^p\right)^{\frac{1}{p}} \leq \|T\|_{\Gamma_p}\left\|(x_j^{**})_{j=1}^n\right\|_p^w \sup_{1\leq j\leq n}\|y_j^*\|.$$

Since $y_1^*, y_2^*, \ldots, y_n^*$ are arbitrary, we get

$$\left(\sum_{j=1}^{n}\|T^{**}x_j^{**}\|^p\right)^{\frac{1}{p}} \leq \|T\|_{\Gamma_p}\left\|(x_j^{**})_{j=1}^n\right\|_p^w.$$

Thus T^{**} is disjoint p-summing and $\|T^{**}\|_{\Gamma_p} \leq \|T\|_{\Gamma_p}$.

The converse is obvious and $\|T\|_{\Gamma_p} \leq \|T^{**}\|_{\Gamma_p}$, just reminding that the canonical $J_X : X \to X^{**}$ is an isometric lattice isomorphism into.

□

5 The maximal properties of the classes of operators

Let us begin this section with recalling the definition of maximal Banach operator ideals.

Definition 5.1 [7] Let $(\mathfrak{A}, \mathbf{A})$ be a Banach operator ideal.

(1) For $T \in \mathcal{L}(E, F)$ define

$$\mathbf{A}^{\max}(T) := \sup\{\mathbf{A}(Q_N T i_M) : M \in FIN(E), N \in COFIN(F)\}$$

$$\mathfrak{A}^{\max}(E,F) := \{T \in \mathcal{L}(E,F) : \mathbf{A}^{\max}(T) < \infty\}$$

and call $(\mathfrak{A}, \mathbf{A})^{\max} := (\mathfrak{A}^{\max}, \mathbf{A}^{\max})$ the maximal hull of $(\mathfrak{A}, \mathbf{A})$.

(2) $(\mathfrak{A}, \mathbf{A})$ is called *maximal* if $(\mathfrak{A}, \mathbf{A}) = (\mathfrak{A}^{\max}, \mathbf{A}^{\max})$.

There is another criterion for the maximal hull $(\mathfrak{A}, \mathbf{A})^{\max}$.

Theorem 5.2 [18] *Let $(\mathfrak{A}, \mathbf{A})$ be a Banach operator ideal. An operator $T \in \mathcal{L}(E,F)$ belongs to $\mathfrak{A}^{\max}(E,F)$ if and only if there exists a constant $C > 0$ such that*

$$\mathbf{A}(RTS) \le C\|R\|\|S\| for all S \in \mathcal{F}(G,E) and R \in \mathcal{F}(F,H),$$

where G, H are arbitrary Banach spaces.

In this case,

$$\mathbf{A}^{\max}(T) = \inf C.$$

Replacing the finite-dimensional subspaces in Definition 5.1 by finite-dimensional sublattices, we introduce the maximal hull of the class of positive p-summing operators in a natural way as follows.

Definition 5.3 For an operator $T : X \to E$, we define

$$\|T\|_{\Lambda_p}^{\max} := \sup\{\|Q_N T i_M\|_{\Lambda_p} : M \in LDim(X), N \in COFIN(E)\}$$
$$\Lambda_p^{\max}(X,E) := \{T \in \mathcal{L}(X,E) : \|T\|_{\Lambda_p}^{\max} < \infty\}.$$

It is easy to see that positive p-summing operators have the right positive ideal property, that is, if $T \in \Lambda_p(X,E)$, $S \in \mathcal{L}(E,F)$ and $R : Y \to X$ is positive, then STR is positive p-summing and $\|STR\|_{\Lambda_p} \le \|S\|\|T\|_{\Lambda_p}\|R\|$. According to this simple fact and Theorem 5.2, the maximal hull of the class of positive p-summing operators can be characterized by the right side composition via positive finite rank operators as expected.

Theorem 5.4 *Let X be an order complete Banach lattice and E be a Banach space. The following statements are equivalent for an operator $T : X \to E$:*

(a) *T is positive p-summing.*
(b) *There exists a constant $C > 0$ such that*

$$\|STR\|_{\Lambda_p} \le C\|S\|\|R\| for all R \in \mathcal{F}(Y,X)_+, S \in \mathcal{F}(E,F),$$

where Y is arbitrary Banach lattice and F is arbitrary Banach space.

(c) *$T \in \Lambda_p^{\max}(X,E)$.*

In this case, $\|T\|_{\Lambda_p} = \inf\{C : as\ in\ (b)\} = \|T\|_{\Lambda_p}^{\max}$.

Proof The implication $(a) \Rightarrow (b)$ is obvious and $\inf\{C : \text{as in } (b)\} \le \|T\|_{\Lambda_p}$. The implication $(b) \Rightarrow (c)$ is trivial and $\|T\|^{\max}_{\Lambda_p} \le \inf\{C : \text{as in } (b)\}$.

Let $T \in \Lambda_p^{\max}(X, E)$. Given any finite families $(x_j)_{j=1}^n$ in X_+. For each $j = 1, 2, \ldots, n$, we choose $u_j^* \in B_{E^*}$ with $\|Tx_j\| = \langle u_j^*, Tx_j\rangle$. Let $M = \operatorname{span}\{x_j : j = 1, 2, \ldots, n\}$ and $N = \bigcap_{j=1}^n \operatorname{Ker}(u_j^*)$. Let $\epsilon > 0$. It follows from Lemma 2.9 that there exist a sublattice Z of X containing M, a finite-dimensional sublattice G of Z and a positive projection P from Z onto G such that $\|Px - x\| \le \epsilon\|x\|$ for all $x \in M$. For each j, we take $\varphi_j \in (E/N)^*$ such that $\|\varphi_j\| = \|u_j^*\|$ and $Q_N^*\varphi_j = u_j^*$. Then one has

$$\begin{aligned}
\left(\sum_{j=1}^n \|Tx_j\|^p\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^n |\langle u_j^*, Tx_j\rangle|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{j=1}^n |\langle u_j^*, T(x_j - Px_j)\rangle|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |\langle u_j^*, TPx_j\rangle|^p\right)^{\frac{1}{p}} \\
&\le \epsilon\|T\| \left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |\langle \varphi_j, Q_N T i_G P x_j\rangle|^p\right)^{\frac{1}{p}} \\
&\le \epsilon\|T\| \left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}} + \|T\|^{\max}_{\Lambda_p} \|(Px_j)_{j=1}^n\|_p^w \\
&\le \epsilon\|T\| \left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}} + \|T\|^{\max}_{\Lambda_p}(1+\epsilon)\|(x_j)_{j=1}^n\|_p^w
\end{aligned}$$

Letting $\epsilon \to 0$, we get

$$\left(\sum_{j=1}^n \|Tx_j\|^p\right)^{\frac{1}{p}} \le \|T\|^{\max}_{\Lambda_p} \|(x_j)_{j=1}^n\|_p^w.$$

This implies that T is positive p-summing and $\|T\|_{\Lambda_p} \le \|T\|^{\max}_{\Lambda_p}$, which completes the proof. □

As to the class of disjoint p-summing operators, we introduce the maximal hull in a similar manner.

Definition 5.5 For an operator $T : X \to E$, we define

$$\begin{aligned}
&\|T\|^{\max}_{\Gamma_p} := \sup\{\|Q_N T i_M\|_{\Gamma_p} : M \in LDim(X), N \in COFIN(E)\} \\
&\Gamma_p^{\max}(X, E) := \{T \in \mathcal{L}(X, E) : \|T\|^{\max}_{\Gamma_p} < \infty\}.
\end{aligned}$$

It easily follows from Definition 4.1 that disjoint p-summing operators have the right disjointness-preserving ideal property, that is, if $T \in \Gamma_p(X, E)$, $S \in \mathcal{L}(E, F)$ and $R : Y \to X$ is disjointness preserving, then STR is disjoint p-summing and $\|STR\|_{\Gamma_p} \leq \|S\|\|T\|_{\Gamma_p}\|R\|$. According to this observation, we characterize the maximal hull of the class of disjoint p-summing operators in terms of disjointness preserving finite rank operators.

Theorem 5.6 *The following statements are equivalent for an operator $T : X \to E$:*

(a) *T is disjoint p-summing.*
(b) *There exists a constant $C > 0$ such that $\|STR\|_{\Gamma_p} \leq C\|S\|\|R\|$ for all $S \in \mathcal{F}(E, F)$ and every disjointness preserving operator $R \in \mathcal{F}(Y, X)$.*
(c) *$T \in \Gamma_p^{\max}(X, E)$.*

In this case, $\|T\|_{\Gamma_p} = \inf\{C : as\ in\ (b)\} = \|T\|_{\Gamma_p}^{\max}$.

Proof $(a) \Rightarrow (b)$ is clear and $\inf\{C : \text{as in } (b)\} \leq \|T\|_{\Gamma_p}$. The implication $(b) \Rightarrow (c)$ is trivial and $\|T\|_{\Gamma_p}^{\max} \leq \inf\{C : \text{as in } (b)\}$.

Let $T \in \Gamma_p^{\max}(X, E)$. Given pairwise disjoint elements $x_1, x_2, \ldots, x_n$ in X. Then $M = \text{span}\{x_1^+, x_2^+, \ldots, x_n^+, x_1^-, x_2^-, \ldots, x_n^-\}$ is a finite dimensional sublattice of X. For each j, we choose $y_j^* \in B_{E^*}$ with $\langle y_j^*, Tx_j\rangle = \|Tx_j\|$. We put $N = \bigcap_{j=1}^n \text{Ker}(y_j^*)$. Note that $Q_N^* : (E/N)^* \to N^\perp$ is a surjective isometry. Let $\varphi_j \in B_{(E/N)^*}$ be such that $Q_N^*(\varphi_j) = y_j^*$ for all $j = 1, 2, \ldots, n$. Then we have

$$\begin{aligned}
\left(\sum_{j=1}^n \|Tx_j\|^p\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^n |\langle y_j^*, Tx_j\rangle|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^n |\langle \varphi_j, Q_N T i_M x_j\rangle|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{j=1}^n \|Q_N T i_M x_j\|^p\right)^{\frac{1}{p}} \\
&\leq \|T\|_{\Gamma_p}^{\max} \|(x_j)_{j=1}^n\|_p^w.
\end{aligned}$$

Hence T is disjoint p-summing and $\|T\|_{\Gamma_p} \leq \|T\|_{\Gamma_p}^{\max}$. □

Evidently, positive p-majorizing operators have the left positive ideal property, that is, if $S \in \Upsilon_p(E, X)$, $T \in \mathcal{L}(F, E)$ and $R : X \to Y$ is positive, then RST is positive p-majorizing and $\|RST\|_{\Upsilon_p} \leq \|R\|\|S\|_{\Upsilon_p}\|T\|$. This elementary observation makes us introduce the maximal hull of the class of p-majorizing operators in the following natural manner.

Definition 5.7 We say that an operator $S : E \to X$ belongs to $\Upsilon_p^{\max}(E, X)$ if there exists a constant $C > 0$ such that $\|RSU\|_{\Upsilon_p} \leq C\|U\|\|R\|$ for all $U \in \mathcal{F}(F, E)$, $R \in \mathcal{F}(X, Y)_+$. We let $\|S\|_{\Upsilon_p}^{\max}$ be the infimum of all such constants C.

The following result suggests that the class of positive p-majorizing operators is maximal in our sense.

Theorem 5.8 $[\Upsilon_p^{\max}, \|\cdot\|_{\Upsilon_p}^{\max}] = [\Upsilon_p, \|\cdot\|_{\Upsilon_p}]$

Proof It is obvious that $\Upsilon_p \subseteq \Upsilon_p^{\max}$ and $\|\cdot\|_{\Upsilon_p}^{\max} \leq \|\cdot\|_{\Upsilon_p}$.

Let $S \in \Upsilon_p^{\max}(E, X)$. Let C be described as in Definition 5.7. By Theorem 2.11, it suffices to prove that S^* is positive p^*-summing. Given any $U \in \mathcal{F}(E^*, F)$ and $R \in \mathcal{F}(Y, X^*)_+$. We may assume that F is finite dimensional. Let $\epsilon > 0$. It follows from [12, Lemma 3.1] that there is a $weak^*$-continuous operator $V : E^* \to F$ such that $U|_{S^*RY} = V|_{S^*RY}$ and $\|V\| \leq (1+\epsilon)\|U\|$. Let $B : F^* \to E$ be an operator such that $B^* = V$. We set $A = R^* J_X$. Then A is positive and finite rank. It follows from the definition of $\Upsilon_p^{\max}(E, X)$ that $\|ASB\|_{\Upsilon_p} \leq C\|A\|\|B\|$. By Theorem 2.11, $B^*S^*A^* = (ASB)^*$ is positive p^*-summing and $\|B^*S^*A^*\|_{\Lambda_{p^*}} \leq C\|A\|\|B\|$. This yields that $B^*S^*A^*J_Y$ is positive p^*-summing. Note that $B^*S^*A^*J_Y = VS^*R = US^*R$. Hence US^*R is positive p^*-summing and satisfies

$$\|US^*R\|_{\Lambda_{p^*}} \leq \|B^*S^*A^*\|_{\Lambda_{p^*}} \leq C\|A\|\|B\| \leq C\|R\|\|V\| \leq C\|R\|(1+\epsilon)\|U\|.$$

Letting $\epsilon \to 0$, we get $\|US^*R\|_{\Lambda_{p^*}} \leq C\|R\|\|U\|$. By Theorem 5.4, we see that S^* is positive p^*-summing and $\|S^*\|_{\Lambda_{p^*}} \leq C$. Hence S is positive p-majorizing and $\|S\|_{\Upsilon_p} \leq \|S\|_{\Upsilon_p}^{\max}$. This completes the proof.

□

It is clear that positive (p, q)-dominated operators have the (two-sided) positive ideal property, that is, if $T \in \Psi_{(p,q)}(X, Y)$, $S : Y \to Z$ is positive and $R : Z \to X$ is positive, then STR is positive (p, q)-dominated and $\|STR\|_{\Psi_{(p,q)}} \leq \|S\|\|T\|_{\Psi_{(p,q)}}\|R\|$. This elementary fact leads to the following definition in a natural way.

Definition 5.9 We say that an operator $T : X \to Y$ belongs to $\Psi_{(p,q)}^{\max}(X, Y)$ if there exists a constant $C > 0$ such that RTS is positive (p, q)-dominated and

$$\|RTS\|_{\Psi_{(p,q)}} \leq C\|R\|\|S\|$$

for all $S \in \mathcal{F}(Z, X)_+$ and $R \in \mathcal{F}(Y, W)_+$.

We put

$$\|T\|_{\Psi_{(p,q)}^{\max}} := \inf C.$$

The following result shows that the class of positive (p, q)-dominated operators is maximal in our sense under the hypothesis of order completeness.

Theorem 5.10 *Let X be an order complete Banach lattice. Then*

$$\left[\Psi_{(p,q)}(X, Y), \|\cdot\|_{\Psi_{(p,q)}}\right] = \left[\Psi_{(p,q)}^{\max}(X, Y), \|\cdot\|_{\Psi_{(p,q)}^{\max}}\right],$$

for every Banach lattice Y.

Proof Obviously, $\Psi_{(p,q)}(X,Y) \subseteq \Psi^{\max}_{(p,q)}(X,Y)$ and $\|\cdot\|_{\Psi^{\max}_{(p,q)}} \leq \|\cdot\|_{\Psi_{(p,q)}}$.

Let $T \in \Psi^{\max}_{(p,q)}(X,Y)$ with constant C. Given $x_1, x_2, \ldots, x_n \in X_+$, $y_1^*, y_2^*, \ldots, y_n^* \in (Y^*)_+$. Let $\epsilon > 0$. By Lemma 2.9, there exist a sublattice Z of X containing $M = \operatorname{span}\{x_j : 1 \leq j \leq n\}$, a finite-dimensional sublattice G of Z and a positive projection P from Z onto G such that

$$\|Px - x\| \leq \epsilon\|x\| \tag{5.1}$$

for all $x \in M$.

Similarly, there exist a sublattice W of Y^* containing $N = \operatorname{span}\{y_j^* : 1 \leq j \leq n\}$, a finite-dimensional sublattice H of W and a positive projection Q from W onto H such that

$$\|Qy^* - y^*\| \leq \epsilon\|y^*\| \tag{5.2}$$

for all $y^* \in N$.

By (5.1), we get

$$\begin{aligned}\left(\sum_{i=1}^n |\langle y_i^*, Tx_i\rangle|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{i=1}^n |\langle y_i^*, T(x_i - Px_i)\rangle|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n |\langle y_i^*, TPx_i\rangle|^r\right)^{\frac{1}{r}}\\ &\leq \epsilon\|T\|\left(\sum_{i=1}^n \|y_i^*\|^r\|x_i\|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n |\langle y_i^*, TPx_i\rangle|^r\right)^{\frac{1}{r}}.\end{aligned} \tag{5.3}$$

By (5.1) and (5.2), we obtain

$$\begin{aligned}\left(\sum_{i=1}^n |\langle y_i^*, TPx_i\rangle|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{i=1}^n |\langle y_i^* - Qy_i^*, TPx_i\rangle|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n |\langle Qy_i^*, TPx_i\rangle|^r\right)^{\frac{1}{r}}\\ &\leq \epsilon(1+\epsilon)\|T\|\left(\sum_{i=1}^n \|y_i^*\|^r\|x_i\|^r\right)^{\frac{1}{r}}\\ &\quad + \left(\sum_{i=1}^n |\langle Qy_i^*, TPx_i\rangle|^r\right)^{\frac{1}{r}}.\end{aligned} \tag{5.4}$$

Note that for each $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}\langle Qy_i^*, TPx_i\rangle &= \langle Qy_i^*, Ti_GPx_i\rangle\\ &= \langle i_HQy_i^*, Ti_GPx_i\rangle\\ &= \langle (i_H^*J_Y)^*J_HQy_i^*, Ti_GPx_i\rangle\\ &= \langle J_HQy_i^*, (i_H^*J_Y)Ti_GPx_i\rangle.\end{aligned}$$

Since P, Q are positive, we see that $Px_i \geq 0$, $J_H Q y_i^* \geq 0$ for each i. It follows from (5.1) and (5.2) that

$$\begin{aligned}\left(\sum_{i=1}^{n} |\langle Q y_i^*, T P x_i\rangle|^r\right)^{\frac{1}{r}} &= \left(\sum_{i=1}^{n} |\langle J_H Q y_i^*, (i_H^* J_Y) T i_G P x_i\rangle|^r\right)^{\frac{1}{r}} \\ &\leq C\|i_G\|\|i_H^* J_Y\|\|(Px_i)_{i=1}^n\|_p^w \|(J_H Q y_i^*)_{i=1}^n\|_q^w \\ &\leq C\|(Px_i)_{i=1}^n\|_p^w \|(Q y_i^*)_{i=1}^n\|_q^w \\ &\leq C(1+\epsilon)\|(x_i)_{i=1}^n\|_p^w (1+\epsilon)\|(y_i^*)_{i=1}^n\|_q^w \end{aligned} \tag{5.5}$$

Combining (5.3), (5.4) with (5.5), we get

$$\begin{aligned}\left(\sum_{i=1}^{n} |\langle y_i^*, T x_i\rangle|^r\right)^{\frac{1}{r}} &\leq \epsilon(2+\epsilon)\|T\| \left(\sum_{i=1}^{n} \|y_i^*\|^r \|x_i\|^r\right)^{\frac{1}{r}} \\ &\quad + C(1+\epsilon)^2 \left\|(x_i)_{i=1}^n\|_p^w \|(y_i^*)_{i=1}^n\right\|_q^w .\end{aligned}$$

Letting $\epsilon \to 0$, we get

$$\left(\sum_{i=1}^{n} |\langle y_i^*, T x_i\rangle|^r\right)^{\frac{1}{r}} \leq C \left\|(x_i)_{i=1}^n\right\|_p^w \left\|(y_i^*)_{i=1}^n\right\|_q^w .$$

Thus T is positive (p,q)-dominated and $\|T\|_{\Psi_{(p,q)}} \leq \|T\|_{\Psi_{(p,q)}^{\max}}$. The proof is completed.

□

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