Pafnuty Chebyshev, Steam Engines, and Polynomials
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1. Pafnuty Chebyshev

Pafnuty Chebyshev was born in the town of Borovsk, Russia (about 100 kilometers southwest of Moscow) in 1821. Not far from his birthplace is a monastery founded by his namesake, Saint Paphnutius of Borovsk, in 1444. In fact the name Paphnutius, which is Coptic for "man of God", has a long and hallowed history, having been attached to a number of well-known personages over the centuries, dating back to the early days of Christianity in Egypt. The name Chebyshev, on the other hand, seems to be harder to trace back to its origins. The Chebyshev family descended from a Tartar military leader named Chebysh, but I am not sure what this name meant in the language of his day. A web page I came across seemed to suggest that Tartar clans of the time were sometimes named after birds, and that "chebysh" in particular meant "chicken".

In any case, male members of the Chebyshev family had for generations had military occupations — but Pafnuty Chebyshev did not follow in their footsteps, because of the fact that one of his legs was shorter than the other, which meant that he had to walk with the aid of a stick. Instead, his family, which was well-off in his youth, paid for him to be educated by illustrious tutors. By the time Chebyshev graduated from Moscow University at age 20, however, his family had fallen on hard times, and he was forced afterwards to make his own living. The academic positions he took did not at first afford him much of a salary, and he had to live very frugally — a habit which he retained for the rest of his life, even after he became able to live more comfortably.

In good times or bad, however, Chebyshev never stinted on money when it came to building models of new kinds of machines, a passion of his from childhood. Among his constructions were calculating machines, one of which still exists in the Musée des Arts et Métiers in Paris, and a "Foot-Stepping Machine" which was a hit at the Paris World's Fair in 1878, and which you can see a nice animation of on YouTube. At the 1893 Columbian World's Exposition in Chicago, Chebyshev displayed seven inventions, one of which was an "armchair velocipede": a sort of wheelchair which could be propelled by levers worked by the arms. In those days women's clothes did not allow them to ride bicycles, and such a device provided them a suitable substitute.

Chebyshev became a successful professor and teacher, and was well-liked and even revered by his students. He encouraged them to visit his home to discuss their work, and they would invariably leave with new ideas and renewed enthusiasm. Among his students and his students' students are several who went on to become famous mathematics in their own right, including Markov and Lyapunov. In fact, he and his students were collectively responsible for turning the city of St. Petersburg into an important center of mathematical learning.

When Chebyshev's finances improved he began to buy land, and kept at it assiduously, so that by the time of his death he was the owner of many estates. He never married, although he did have a daughter whom he supported financially. He lived alone in a large house, carefully locking the maid out of his rooms at night. He retired from his professorship at age 61, but continued to work on mathematics right up to his death at age 73.

2. Steam Engines

Chebyshev was foremost what we would call today a pure mathematician. Among his contributions to pure mathematics was a landmark result on the distribution of prime numbers, showing that if the limit of $\pi(n)\log(n)/n$ exists as $n \to \infty$, then the limit must be 1. (He did not, however, prove that the limit existed; this was not done till some years afterwards by others.) He also did important work in probability theory and on questions associated with the possibility of finding integrals in finite terms for certain classes of functions.

However, he always maintained a strong interest in the applications of mathematics. He had a gift for discovering beautiful mathematics in seemingly mundane settings and distilling it into theory which reached far beyond the initial application.

One of his main interests was the theory of mechanisms called linkages, used in steam engines and other machines to convert one type of motion into another. A famous example of his time was a linkage devised by James Watt for converting the rotational motion of the rocker beam in a pumping engine into the rectilinear motion of a piston shaft. Chebyshev aimed to find mathematical methods for systematically devising linkages to produce desired types of motion with high accuracy.

Watt's linkage was constructed so that a certain point on the linkage would move almost in a straight line, but not quite in a straight line, deviating from it by about 1 part in 4000. One of Chebyshev early goals was to find a linkage that would either produce exact straight-line motion, or motion which was even closer to a straight line than Watt's linkage produced. He did not succeed in producing exact straight-line motion, but in the 1850's he did come up with a linkage that produced motion that deviated from a straight line by only one part in 8000.

Eventually, Chebyshev's student Lipkin found a linkage that produced exact straight-line motion. (Later a French army officer, Peaucellier, came forward with the news that he had already discovered the same linkage, without giving the details in print. In fact, the solution, once you see it, is simple enough that it is quite likely that it could be discovered by different people independently.) However, the Lipkin-Peaucellier linkage wasn't so useful as a practical device: it was made of so many bars (seven) and joints (six) that the errors involved in constructing it would accumulate to the point that they would wipe out the theoretical advantage gained.

Chebyshev's work on linkages was not restricted to the straight-line problem: by the end of his life he had written many papers on different linkages and their properties, and had used a number of clever linkages in the machines he constructed.

3. Polynomials

How exactly did Chebyshev arrive at the improvements he was able to make in the linkages and mechanisms used at the time? I read his first published paper on the subject, "The theory of mechanisms that are known by the name of parallelograms" (1854), in an attempt to find this out. In this paper Chebyshev describes a new method for approximating functions by polynomials, and gives a couple of results concerning Watt linkages that he says he derived through the use of his method. But the part of the paper that explains how these results are derived is missing: the paper ends abruptly with the promise that the explanation is in the "next sections", which do not exist. Chebyshev later wrote that he did not have time to finish writing the paper. One suspects that he concluded that these particular results were not of great practical interest after all. Nevertheless, the theoretical problem Chebyshev considered in this paper turned out to have practical applications reaching far beyond the problem of designing linkages.

Here is my guess at some of what may have going through Chebyshev's mind when he wrote his 1854 paper. He begins with the observation that in the motion of a piston shaft in a steam engine, it is important to know its maximum deviation from a straight line. (Such deviations, for example, cause leaks in the "stuffing box" — an important part of certain engines to this day — which degrade the performance of the engine.) We would like to make this maximum deviation as small as possible. Thus, if the horizontal motion of the shaft is between $x = -1$ and $x = 1$, and its deviation is given by the curve $y = f(x)$ in the xy-plane, then we would like to minimize the quantity

$$\|f\| = \sup_{-1 \le x \le 1} |f(x)|.$$

To do this, we can first approximate f by its Taylor polynomial at $x = 0$, thus effectively replacing f by a polynomial, of degree n say:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where the coefficients a_i are real numbers. We assume we can give the coefficients of f almost any values we desire, by changing the configuration of the linkage. But there will be some constraint on the coefficients, or otherwise we could simply take all them equal to zero. As a simple example of such a constraint, let us assume that we cannot change the coefficient of the highest term x^n, and are required to take $a_n = 1$. Then the question becomes: how can we choose the coefficients a_0, a_1, ... , a_{n-1} so that if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + x^n,$$

then $\|f\|$ is as small as possible?

This is exactly the question Chebyshev winds up considering in his 1854 paper. His solution depends on the fact that in order to minimize $\|f\|$ we need to choose f so that $|f|$ takes its maximum value exactly $n+1$ times, with alternating signs. Chebyshev does not prove this fact, or say how he knows this to be true. (Some suggest that he learned it from Poncelet.) Let us give a proof here:

Suppose f is a polynomial of degree n, with $a_n = 1$, and $|f|$ takes its maximum value $n+1$ times, with alternating signs. We want to show that if p is any polynomial of the form

$$p(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} + x^n,$$

then we cannot have $\|p\| < \|f\|$. To see this, we suppose $\|p\| < \|f\|$ and argue by contradiction. From $\|p\| < \|f\|$ it follows that $|p(x_i)| < 1$ for $i = 1, \ldots, n+1$. Let $h = p - f$; then h is a polynomial of degree $n-1$, because the terms x^n in p and f cancel when subtracted. But $|h(x_i)| > 0$ for $i = 1, \ldots, n+1$, and since f alternates in sign as we move through x_1, ... , x_{n+1}, then so does h. (A sketch of the graphs of p and f makes this clearer.) From the Intermediate Value Theorem it then follows that h has at least n zeroes, one in each interval (x_i, x_{i+1}) for $i = 1, \ldots, n$. But since h is a polynomial of degree $n-1$ it then follows that h must be identically zero. Therefore $p \equiv f$, which contradicts $\|p\| < \|f\|$, and we are done.

We have now reduced the problem of finding the minimizing polynomial f to the problem of finding a polynomial of degree n which takes its maximum absolute value on $[-1, 1]$ exactly $n+1$ times, with alternating signs. But such a polynomial f is given as follows: first observe that $g(\theta) = \cos(n\theta)$ takes its maximum absolute value of 1 exactly $n+1$ times on $0 \le \theta \le \pi$, with alternating signs; then observe that if we make the change of variables $x = \cos\theta$, g becomes a polynomial in x with the desired properties on $-1 \le x \le 1$. That is, the desired polynomial is given concisely by defining

$$T_n(x) = \cos(n(\cos^{-1} x)),$$

where $\cos^{-1} x$ maps $[-1, 1]$ onto $[0, \pi]$, and letting f be the polynomial obtained by dividing T_n by the correct constant. That is, if we write T_n as

$$T_n = t_0 + t_1 x + \cdots + t_{n-1} x^{n-1} + t_n x^n,$$

then f should equal T_n / t_n. It is easy to see (see below) that $t_n = 2^{n-1}$, so we can write f explicitly as

$$f = \frac{1}{2^{n-1}} T_n.$$

The polynomials T_n are known as Chebyshev polynomials. Let us say a little more about some of their remarkable properties. As mentioned above, for each $n \in \mathbf{N}$, we let $x = \cos\theta$, and then define $T_n(x)$ by

$$T_n(x) = \cos(n\theta).$$

You can check that this defines T_n uniquely as a polynomial of degree n. For example, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$.

An easy way to compute Chebyshev polynomials is by recursion, starting from $T_0(x) = 1$ and $T_1(x) = x$, and then using the recursion relation

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x),$$

which is readily proved from the definition.

For each integer n, let P_n stand for the space of polynomials of degree n with real coefficients. Let B_n stand for the set of polynomials $p \in P_n$ which satisfy $\|p\| \leq 1$, where the norm $\|p\|$ of p is defined as above by

$$\|p\| = \sup_{-1 \leq x \leq 1} |p(x)|.$$

If $p \in P_n$, we define the "leading coefficient" of p to be the coefficient of x^n in p (by this definition, the leading coefficient may equal 0). From the recursion relation for T_n we now see by induction that the leading coefficient of T_n is 2^{n-1}.

What we proved above was most of the following theorem:

Theorem: If we let $\tilde{T}_n = T_n/2^{n-1}$, then $\tilde{T}_n$ has norm less than that of any other polynomial $p \in P_n$ with leading coefficient 1.

But we did not prove this theorem completely. We did show that no polynomial p in P_n with leading coefficient 1 can have norm less than that of $\tilde{T}_n$, which is 1; what remains to be proved is that $\tilde{T}_n$ is the *only* polynomial in P_n with leading coefficient 1 whose norm is equal to 1. This can be done by a straightforward tightening of the argument given above. (See Rivlin's book, referenced below, for details.)

The above theorem is related to the following interesting fact: if $p \in P_n$ and $|p|$ attains its maximum value on $[-1, 1]$ at $n+1$ distinct points, then either p is a constant, or p is a multiple of T_n.

Here are some more extremal properties of Chebyshev polynomials, taken from Rivlin's book, where they are all proved. (In what follows we use $p^{(k)}$ to denote the kth derivative of p and $T_n^{(k)}$ to denote the kth derivative of T_n.)

- The closest polynomial in P_{n-1} to x^n (when distance is measured in the norm defined above) is $x^n - \tilde{T}_n$.
- The largest value of $|p^{(n)}(0)|$ attained for $p \in B_n$ is $2^{n-1} \cdot n!$, and this is attained only by $p = \pm T_n$.
- If $p \in B_n$ then $p(x) \leq T_n(x)$ for all $x > 1$, and if $p(x) = T_n(x)$ for some $x > 1$ then $p(x) = T_n(x)$ for all x. The same results hold for p', p'', etc., up to the nth derivative.
- Suppose the coefficients of $T_n(x)$ are given by

  $$T_n(x) = t_0 + t_1 x + t_2 x^2 + \cdots + t_n x^n,$$

  and suppose k is any integer such that $0 \leq k \leq n$ and $n - k$ is even (so that t_k is non-zero). Then for every polynomial $p \in B_n$, if we write

  $$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

  we have $|a_k| \leq |t_k|$. Moreover, equality holds (for $k > 0$ and $n > 2$) only if $p = \pm T_n$.
- If $p \in B_n$ then $\|p'\| \leq n^2$, with equality only for $p = \pm T_n$. This was first proved by A. A. Markov, a student of Tchebyshev's.
- If $p \in B_n$ then $\|p^{(k)}\| \leq |T_n^{(k)}(1)|$ for all $k \geq 1$. This was first proved by V. A. Markov, another student of Tchebyshev's. The proof is decidedly not easy.

Finally, Rivlin also mentions one other curious extremal property of Chebyshev polynomials, but refers the reader elsewhere for the proof. Erdös proved in 1939 that among all trigonometric polynomials τ of degree n (i.e., linear combinations of $\sin\theta$, $\cos\theta$, $\sin 2\theta$, $\cos 2\theta$, $\ldots$, $\sin n\theta$, $\cos n\theta$) and with norm $\|\tau\| = \sup_{0 \leq \theta \leq \pi} |\tau(\theta)|$ satisfying $\|\tau\| \leq 1$, the one whose graph has the longest arc length on the interval $0 \leq \theta \leq \pi$ is given by $\tau(\theta) = \cos n\theta$. This fact led him to conjecture that among all polynomials p in B_n, the one whose graph has the longest arc length on the interval $-1 \leq x \leq 1$ is T_n. This turns out to be true, but it was not proved till 40 years later. Two mathematicians, Kristiansen and Bojanov, gave proofs independently; their proofs are quite hard.

The above discussion focuses on just one aspect of Chebyshev polynomials: namely, their extremal properties, which are what led Chebyshev to study them in the first place. Later Chebyshev came to realize that there is a quite different, yet equally important, way to approach these polynomials: namely, to view them as orthogonal polynomials with respect to a certain inner product defined on a vector space of functions from $[-1, 1]$ to $\mathbf{R}$. Chebyshev's appreciation of the latter fact led him to be the first to investigate orthogonal polynomials in full generality; today this is a vast subject which remains an active topic of research.

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