EXAM 1
Math 1914
9-22-14

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* Give a proof of the product rule for differentiation.

Suppose $y = uv$, where u and v are functions of x. Suppose x increases by an amount Δx, and $\Delta u, \Delta v, \Delta y$ are the corresponding increases in u, v, and y. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u+\Delta u)(v+\Delta v) - uv}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{uv} + \Delta u \cdot v + u \cdot \Delta v + \Delta u \cdot \Delta v - \cancel{uv}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot v + u \cdot \frac{\Delta v}{\Delta x} + \Delta u \cdot \frac{\Delta v}{\Delta x} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} + 0 \cdot \frac{dv}{dx}$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}.$$

2. *(14 points)* Use the definition of derivative to calculate $f'(a)$, where $f(x) = \dfrac{1}{\sqrt{x}}$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{x \to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x-a} = \lim_{x \to a} \frac{\left(\frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}}\right)}{x-a} =$$

$$= \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})}{\sqrt{x}\sqrt{a}\,(x-a)} = \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{x}\sqrt{a}\,(x-a)(\sqrt{a}+\sqrt{x})}$$

$$= \lim_{x \to a} \frac{a - x}{\sqrt{x}\sqrt{a}\,(x-a)(\sqrt{a}+\sqrt{x})} = \lim_{x \to a} \frac{(-1)\cancel{(x-a)}}{\sqrt{x}\sqrt{a}\,\cancel{(x-a)}(\sqrt{a}+\sqrt{x})} =$$

$$= \lim_{x \to a} \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})} = \frac{-1}{\sqrt{a}\sqrt{a}(\sqrt{a}+\sqrt{a})} = \frac{-1}{\sqrt{a}\sqrt{a}(2\sqrt{a})}$$

$$= \frac{-1}{2(\sqrt{a})^3}.$$

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3. (~~24~~ 22 *points)* Find the derivative, showing all work. (In this and all the subsequent problems on the test, you may use any of the rules of differentiation.)

[6] a. $y = \dfrac{\sqrt{x}-x}{x^3+1}$ $= \dfrac{u}{v}$; by the Quotient Rule, with $u = \sqrt{x} - x$ and $v = x^3+1$

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{(x^3+1)\left(\frac{1}{2}x^{-\frac{1}{2}} - 1\right) - (\sqrt{x}-x)(3x^2)}{(x^3+1)^2}$$

(1) (1) (1) (3)

[8] b. $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$ $= \dfrac{4}{\cos x} + \dfrac{\cos x}{\sin x}$ (2). You can use the Quotient Rule:

$$\frac{dy}{dx} = \frac{\cos x \cdot \frac{d}{dx}(4) - 4 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} + \frac{\sin x\,\frac{d}{dx}(\cos x) - \cos x\,\frac{d}{dx}(\sin x)}{\sin^2 x}$$

(2) (2)

$$= \frac{0 + 4\sin x}{\cos^2 x} + \frac{\sin x \cdot (-\sin x) - \cos x\,(\cos x)}{\sin^2 x}.$$

(2) (Other methods are possible)

[8] c. $y = x^3 \sin x \cos x$

Let $u = x^3$ and $v = \sin x \cos x$. Then $\frac{du}{dx} = 3x^2$ and

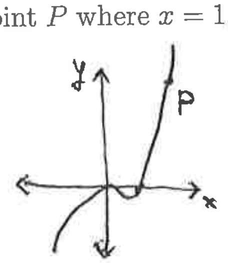
by the Product Rule, $\frac{dv}{dx} = \sin x \cdot (-\sin x) + \cos x \cdot \cos x$, so (2) (2)

$\frac{dv}{dx} = -\sin^2 x + \cos^2 x$. By the Product Rule again,

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx} = \boxed{x^3\left(-\sin^2 x + \cos^2 x\right) + \sin x \cos x \cdot 3x^2}$$

(2) (2)

4. *(10 points)* Find the equation of the tangent line to the graph of $y = 5x^3 - 2x^2$ at the point P where $x = 1$.



The slope of the tangent line at P (2)

is $\left.\frac{dy}{dx}\right|_{x=1} = 15x^2 - 4x\,\Big|_{x=1} = 15 - 4 = 11.$ (2)

The coordinates of P are $(x, 5x^3 - 2x^2)\Big|_{x=1} = (1, 3).$ (2)

The equation of the tangent line is

$$(y-3) = 11(x-1).$$

(2) (2)

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5. *(10 points)* Use the precise definition of limit to show that $\lim_{x \to 4} 1 + 3x = 13$.

Let $\varepsilon > 0$ be given. (2)

Choose $\delta = \frac{\varepsilon}{3}$. (2)

If $-\delta < x - 4 < \delta$, then (2)

$-\frac{\varepsilon}{3} < x - 4 < \frac{\varepsilon}{3}$,

So $-\varepsilon < 3x - 12 < \varepsilon$, (2)

So $-\varepsilon < (1+3x) - 13 < \varepsilon$. (2)

6. *(16 points)* Evaluate the limit, showing all work. (2)

[6] a. $\lim_{x \to -3} \frac{x+3}{x^2+4x+3} = \lim_{x \to -3} \frac{x+3}{(x+3)(x+1)}$ (2) $= \lim_{x \to -3} \frac{1}{x+1} =$

$= \frac{1}{-3+1} = \frac{1}{-2} = -\frac{1}{2}$. (2)

[10] b. $\lim_{x \to 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{x} = \lim_{x \to 0} \frac{\left[\frac{(x+1)+(x-1)}{(x-1)(x+1)}\right]}{x}$ (2) $=$

$= \lim_{x \to 0} \frac{\left(\frac{2x}{(x-1)(x+1)}\right)}{x}$ (2) $= \lim_{x \to 0} \frac{2x}{x(x-1)(x+1)}$ (2)

$= \lim_{x \to 0} \frac{2}{(x-1)(x+1)}$ (2) $= \frac{2}{(-1)(1)} = -2$. (2)

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7. *(18 points)* An object moves along the x-axis. Its position at time t is given by

$$x = \frac{25}{t^2} - \frac{5}{t}.$$

a. Find the object's average velocity over the time interval between $t = 1$ and $t = 5$.

[6] At $t=1$, $x = \frac{25}{1} - \frac{5}{1} = 20$. 1

At $t=5$, $x = \frac{25}{25} - \frac{5}{5} = 1 - 1 = 0$. 1

So the change in x on this time interval is $\Delta x = 0 - 20 = -20$. 2
The change in t is $\Delta t = 5 - 1 = 4$. So the average velocity is $\frac{\Delta x}{\Delta t} = \frac{-20}{4} = -5$. 2

b. Find the object's (instantaneous) velocity at time $t = 1$.

[6]

It is $\left.\frac{dx}{dt}\right|_{t=1}$ 2 $= \left.\frac{d}{dt}\left(25t^{-2} - 5t^{-1}\right)\right|_{t=1} =$

$= \left.\left(-50t^{-3} + 5t^{-2}\right)\right|_{t=1}$ 2 $= -50 + 5 = -45$. 2

[6] **c.** Find the object's acceleration at time $t = 1$.

It is $\left.\frac{d^2x}{dt^2}\right|_{t=1}$ 2 $= \left.\frac{d}{dt}\left(-50t^{-3} + 5t^{-2}\right)\right|_{t=1} =$

2

$= \left.\left(150t^{-4} - 10t^{-3}\right)\right|_{t=1}$

$= 150 - 10 = 140$. 2