EXAM 2
Math 1914
10-27-14

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(15 points)* Suppose a curve is given by the equation

$$\sin(y) + xy = x^3 - 1.$$

a. Find dy/dx by implicit differentiation.

[12]

$$\frac{d}{dx}(\sin y + xy) = \frac{d}{dx}(x^3 - 1) \quad 2$$

$$\cos y \cdot \frac{dy}{dx} \; 2 + x \cdot \frac{dy}{dx} + 1 \cdot y \; 2 = 3x^2 - 0 \; 2$$

$$\frac{dy}{dx}(\cos y + x) = 3x^2 - y \quad 2$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2 - y}{\cos y + x}.} \quad 2$$

b. Find the slope of the curve at the point $(1, 0)$.

[3]

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=0}} = \frac{3 \cdot 1^2 - 0}{\cos 0 + 1} = \frac{3}{1+1} = \boxed{\frac{3}{2}.} \quad 3$$

2. *(15 points)* If $y^3 + y = x^2$, use implicit differentiation to find $\frac{d^2y}{dx^2}$ in terms of x and y. You need not simplify your answer.

$$\frac{d}{dx}(y^3 + y) = \frac{d}{dx}(x^2) \; 2 \Rightarrow 3y^2 \frac{dy}{dx} \; 2 + \frac{dy}{dx} \; 1 = 2x \; 1 \Rightarrow$$

$$\Rightarrow (3y^2 + 1)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{3y^2+1}. \quad 2$$

$$\text{So} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2x}{3y^2+1}\right) \; 2 = \frac{(3y^2+1) \cdot 2 - 2x\left(6y\frac{dy}{dx}\right)}{(3y^2+1)^2} \quad 3$$

$$\text{So} \quad \boxed{\frac{d^2y}{dx^2} = \frac{(3y^2+1) \cdot 2 - 2x \cdot 6y \cdot \left(\frac{2x}{3y^2+1}\right)}{(3y^2+1)^2}} \quad 2$$

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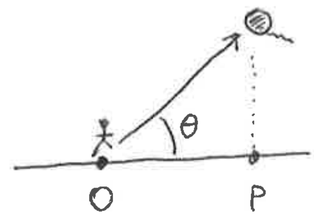
3. *(12 points)* A balloon starts at point P, 50 feet away from an observer at O, and rises at a rate of 3 feet per second. The angle θ is formed between the line of sight of the observer and the ground (see diagram). At what rate is θ changing when the balloon is 50 feet high?



Let y be the height of the balloon. (2)

Since $OP = 50$, then $\tan\theta = \frac{y}{50}$. (2)

So $\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{50}\right)$, and $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50} \cdot \frac{dy}{dt}$. (2) (2)

Thus $\frac{d\theta}{dt} = \frac{1}{\sec^2\theta} \cdot \frac{1}{50} \cdot \frac{dy}{dt} = \frac{\cos^2\theta}{50} \cdot \frac{dy}{dt}$. When $y = 50$, (2)

The hypotenuse of $\triangle OPQ$ is $\sqrt{2}\cdot 50$, so $\cos\theta = \frac{1}{\sqrt{2}}$ and $\cos^2\theta = \frac{1}{2}$.

So $\boxed{\frac{d\theta}{dt} = \frac{(1/2)}{50} \cdot 3 = \frac{3}{100}}$ (radians per second) (2)

4. *(10 points)* For the function $f(x) = x^3 + 3x^2 - 1$,

a. Find the critical numbers.

$\frac{dy}{dx} = 3x^2 + 6x$ (2) $= 3x(x+2)$ (2), so $\frac{dy}{dx} = 0$ at

$\boxed{x = 0 \text{ and } x = -2.}$ (2)

b. Find the maximum and minimum values of the function on the interval $[-10, 0]$.

The values at the endpoints are $f(-10) = -1000 + 300 - 1 = -701$ and $f(0) = -1$. The value at the critical point $x = -2$ is (2)

$f(-2) = -8 + 3\cdot 4 - 1 = 3$.

So the max value is 3 (at $x = -2$) and the min value is -701 (at $x = -10$). (2)

5. *(8 points)* Use Rolle's theorem to prove that the equation $x^5 + 4x - 1 = 0$ cannot have more than one solution.

We prove this by assuming there are two different solutions, at $x = a$ and $x = b$, and showing that this assumption leads to a logical contradiction. (2)

If there are two solutions $x = a$ and $x = b$, then by Rolle's theorem there is some point c in between where $f'(c) = 0$. (2)

This implies that $5c^4 + 4 = 0$. (2) But $5c^4 \geq 0$, and a non-negative number plus 4 can't equal 0. (2) This is the desired contradiction.

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6. *(20 points)* For the function $f(x) = x^3 - 3x^2 - 9x$, find the following, showing all work:

a. Critical number(s) $x=-1,\ x=3$

b. Interval(s) of increase $(-\infty,-1]$ and $[3,\infty)$

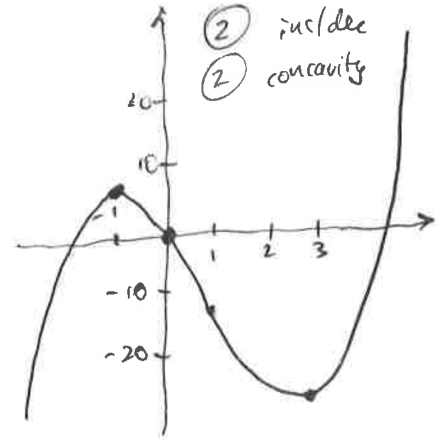
c. Interval(s) of decrease $[-1,3]$

d. Interval(s) where concave up $[1,\infty)$

e. Interval(s) where concave down $(-\infty,1]$

f. Inflection point(s) $x=1$.

g. Use the above information to sketch the graph.



Here $y = x^3-3x^2-9x$,
$y' = 3x^2-6x-9$, and 2
$y'' = 6x-6$. 2

So $y' = 3(x^2-2x-3) = 3(x-3)(x+1)$ 2, and the critical numbers are $x=3$ and $x=-1$. 2

Checking the sign of y':

	-1		3	x
$(x-3)$	$-$	$-$	$+$	
$(x+1)$	$-$	$+$	$+$	
y'	$+$	$-$	$+$	

So $y'>0$ on $(-\infty,-1)$ and $(3,\infty)$: intervals of increase 2 are $(-\infty,-1]$ and $[3,\infty)$
$y'<0$ on $(-1,3)$. : interval of decrease is $[-1,3]$ 2

Also
$y''=6(x-1)$, which gives $y''>0$ for $x>1$ 1 and $y''<0$ for $x<1$.
So f is concave up 1 on $[1,\infty)$ and concave down on $(-\infty,1]$; 1
and $x=1$ is an inflection point 1

Plotting y at $x=-1,1,3$ gives:

x	y
-1	$-1-3+9=5$
1	$1-3-9=-11$
3	$27-27-27=-27$

, and $y=0$ at $x=0$, so the graph is as shown above ↑

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7. *(20 points)* For the function $f(x) = x + \dfrac{4}{x-5}$, find the following if they exist, showing all work:

a. Vertical asymptote(s) $x = 5$ (2)

b. Horizontal asymptote(s) none (2)

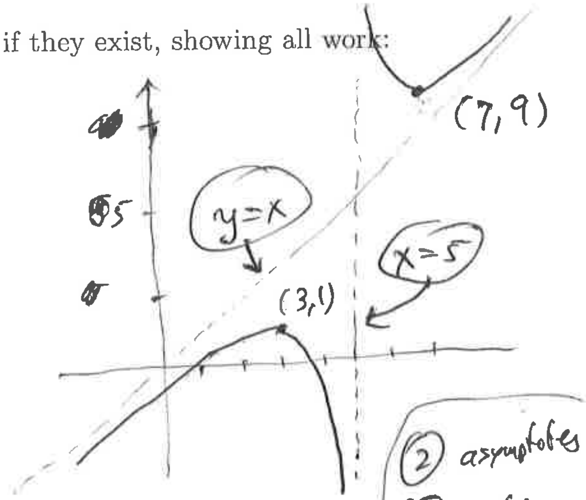
c. Slant asymptote(s) $y = x$ (2)

d. Critical number(s) $x = 3$ and $x = 7$

e. Interval(s) of increase $(-\infty, 3)$ and $(7, \infty)$ (2)

f. Interval(s) of decrease $(3,5)$ and $(5,7)$ (2)

g. Use the above information to sketch the graph.



(2) asymptotes
(2) inc/dec

The only (finite) point where the limit of $f(x)$ is $\pm\infty$ is at $x=5$, so $x=5$ is the vertical asymptote.

As $x \to \infty$ or $x \to -\infty$, the limit of $f(x)$ is not finite, so there is no horizontal asymptote.

Since $\lim\limits_{x\to\infty} f(x) - x = \lim\limits_{x\to\infty} \dfrac{4}{x-5} = 0$

and $\lim\limits_{x\to-\infty} f(x) - x = \lim\limits_{x\to-\infty} \dfrac{4}{x-5} = 0$,

Then the line $y = x$ is a slant asymptote on both the left and the right.

We have $y' = 1 - \dfrac{4}{(x-5)^2}$ (2) $= \dfrac{(x-5)^2 - 4}{(x-5)^2}$ (2) $= \dfrac{x^2 - 10x + 21}{(x-5)^2}$

So $y' = \dfrac{(x-3)(x-7)}{(x-5)^2}$. Thus the points in the domain where $y' = 0$ are $x = 3$ and $x = 7$ (2). These are the critical numbers. (According to the text's definition, $x = 5$ is not a critical number as it is not in the domain.)

We see that $y' > 0$ on $(-\infty, 3)$ and $(7, \infty)$, and $y' < 0$ on $(3,5)$ and $(5,7)$. So the graph is as above ↑

x	y
3	$3 + \frac{4}{-2} = 3 - 2 = 1$
7	$7 + \frac{4}{2} = 7 + 2 = 9$

		3		5		7	
$x-3$	−		+		+		+
$x-7$	−		−		−		+
$(x-5)^2$	+		+		+		+
y'	+		−		−		+