FINAL EXAM Name key

Math 1914

12-10-14

<u>**Instructions**</u> *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* Use the definition of derivative to calculate $f'(a)$ when $f(x) = \dfrac{3}{x} + 5x$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\left[\frac{3}{a+h} + 5(a+h)\right] - \left[\frac{3}{a} + 5a\right]}{h} =$$

$$= \lim_{h \to 0} \left[\frac{\frac{3}{a+h} - \frac{3}{a}}{h} + \frac{5a + 5h - 5a}{h} \right] = \lim_{h \to 0} \left[\frac{\left(\frac{3a - 3(a+h)}{(a+h)a} \right)}{h} + \frac{5h}{h} \right] =$$

$$= \lim_{h \to 0} \left[\frac{3a - 3a - 3h}{(a+h)ah} + 5 \right] = \lim_{h \to 0} \left[\frac{-3h}{(a+h)ah} + 5 \right]$$

$$= \lim_{h \to 0} \left[\frac{-3}{(a+h)a} + 5 \right] = \boxed{-\frac{3}{a^2} + 5}$$

2. *(12 points)* Find the derivative of $y = \dfrac{\sin x}{\cos(x^2) + x}$. (In this and all the subsequent problems on the test, you may use any of the rules of differentiation.)

By the quotient rule, $$\frac{dy}{dx} = \frac{(\cos(x^2) + x) \cdot \cos x - \sin x \frac{d}{dx}(\cos(x^2) + x)}{(\cos(x^2) + x)^2}$$

By the chain rule, $\frac{d}{dx}(\cos(x^2)) = -\sin(x^2) \cdot 2x$. So

$$\frac{dy}{dx} = \boxed{\frac{(\cos(x^2) + x) \cdot \cos x - \sin x \left(-\sin(x^2) \cdot 2x + 1 \right)}{(\cos(x^2) + x)^2}}$$

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3. *(12 points)* Use the precise definition of limit to prove that $\lim_{x\to 3} 2x - 5 = 1$.

Let $\varepsilon > 0$ be given. (2)

Choose $\delta = \varepsilon/2$. (2)

If $-\delta < x - 3 < \delta$ (2) (and $x \neq 3$),

Then $-\frac{\varepsilon}{2} < x - 3 < \frac{\varepsilon}{2}$, (2) so

$-\varepsilon < 2x - 6 < \varepsilon$, (2) and hence

$-\varepsilon < (2x-5) - 1 < \varepsilon$. (2)

4. *(20 points)* Suppose a curve is given by the equation $x^2y^3 + 7x^4y = 22$.

a. Find dy/dx by implicit differentiation.

$$\frac{d}{dx}\left(x^2y^3 + 7x^4y\right) = \frac{d}{dx}(22) \quad (2)$$

$$\Rightarrow 2xy^3 + x^2\, 3y^2 \frac{dy}{dx} + 28x^3y + 7x^4\frac{dy}{dx} = 0 \quad (2)\ (2)\ (2)\ (2)\ (2)$$

$$\Rightarrow \left(3x^2y^2 + 7x^4\right)\frac{dy}{dx} = -\left(2xy^3 + 28x^3y\right) \quad (2)\ (2)\ (2)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-\left(2xy^3 + 28x^3y\right)}{\left(3x^2y^2 + 7x^4\right)}}$$

b. Find the slope of the curve at the point $(1,2)$.

When $x = 1$ and $y = 2$

$$\frac{dy}{dx} = \frac{-(2\cdot 8 + 56)}{(3\cdot 4 + 7)} \quad (2) \quad = \boxed{\frac{-72}{19}}$$

5. *(20 points)* A ball of ice 10 inches in diameter is melting so that its volume decreases at a rate of 3 cubic inches per minute. Recall that for a ball of radius r, the volume is given by $V = \dfrac{4\pi r^3}{3}$ and the surface area is given by $A = 4\pi r^2$.

a. How fast is the radius of the ball decreasing? (2) (2)

[12] (2) $\dfrac{dV}{dt} = \dfrac{d}{dt}\left(\dfrac{4\pi r^3}{3}\right) = \dfrac{4\pi}{3}\cdot 3r^2 \dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt}$; so

$-3 = 4\pi \cdot 5^2 \cdot \dfrac{dr}{dt}$, so $\boxed{\dfrac{dr}{dt} = \left(\dfrac{-3}{100\pi}\right) \text{ inches per minute.}}$

(2) (2) (2)

[8] **b.** How fast is the surface area of the ball decreasing? (2) (2)

(2) $\dfrac{dA}{dt} = \dfrac{d}{dt}(4\pi r^2) = 4\pi \cdot 2r\dfrac{dr}{dt} =$

$= 4\pi \cdot 2 \cdot 5 \cdot \left(\dfrac{-3}{100\pi}\right) = \dfrac{-120\pi}{100\pi} = -\dfrac{6}{5}$ $\boxed{\text{square inches per minute}}$

(2)

6. *(16 points)* Find the absolute maximum and minimum values of the function $f(x) = 2x^5 - 5x^4 = x^4(2x-5)$ on the interval $1 \le x \le 10$. Show all work.

(2) (2)

The critical points of $f(x)$ are where $f'(x) = 10x^4 - 20x^3 = 0$,

(2)

so they are where $10x^3(x-2) = 0$, or at $x = 0$ and $x = 2$. The only one in the interval $[1, 10]$ is at $x = 2$. (2)

(2)

We have $f(1) = 2 - 5 = -3$,

(2)

$f(2) = 2^4(2\cdot 2 - 5) = 16(-1) = -16$, and

$f(10) = 10^4(2\cdot 10 - 5) = (10{,}000)\cdot 15 = 150{,}000$ (2)

$\boxed{\text{So the maximum is } 150{,}000 \text{ at } x = 10, \text{ and the minimum is } -16 \text{ at } x = 2.}$ (2)

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7. *(32 points)* For the function $f(x) = \dfrac{3x}{x^2+4}$, find the following if they exist, showing all work:

a. Vertical asymptote(s) ___None.___ (2)

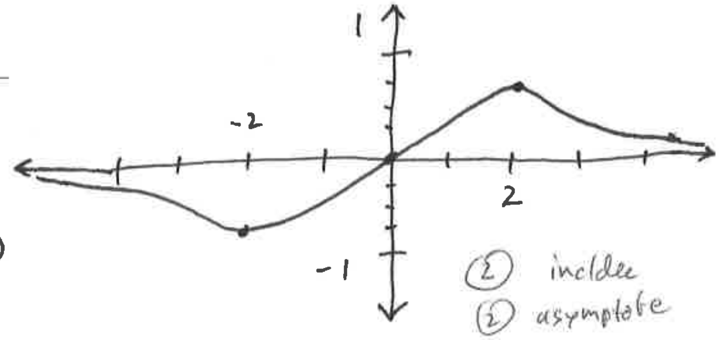
b. Horizontal asymptote(s) ___$y=0$___ (2)

c. Critical number(s) ___$x=-2,\ x=2$___ (2)

d. Interval(s) of increase ___$(-2,2)$___ (2)

e. Interval(s) of decrease ___$(-\infty,-2)$ and $(2,\infty)$___ (2) (2)

f. Use the above information to sketch the graph.



(2) include
(2) asymptote

a.) None, since $x^2+4 \neq 0$ for all $x \in \mathbb{R}$. (2)

b.) $$\lim_{x\to\infty} \frac{3x}{x^2+4} = \lim_{x\to\infty} \frac{\left(\frac{3x}{x^2}\right)}{\left(\frac{x^2+4}{x^2}\right)} = \lim_{x\to\infty} \frac{\left(\frac{3}{x}\right)}{\left(1+\frac{4}{x^2}\right)} = \frac{0}{1+0} = 0,$$ (2)

and similarly $\lim_{x\to-\infty} \dfrac{3x}{x^2+4} = 0$. So $y=0$ is a horiz. asymptote. (2)

c.) $$f'(x) = \frac{(x^2+4)\cdot 3 - 3x(2x)}{(x^2+4)^2} = \frac{3x^2+12-6x^2}{(x^2+4)^2} = \frac{12-3x^2}{(x^2+4)^2}$$ (2)

$$= \frac{3(4-x^2)}{(x^2+4)^2} = \frac{(-3)(x-2)(x+2)}{(x^2+4)^2} = 0$$ (2) when $x=2$ or $x=-2$

d.,
e.) From the table at right ⟶ we see that
$f'(x)<0$ when $-\infty < x < -2$, (2)
$f'(x)>0$ when $-2 < x < 2$, (2)
$f'(x)<0$ when $2 < x < \infty$ (2)

	$x<-2$	$-2<x<2$	$x>2$
$x-2$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
-3	$-$	$-$	$-$
$(x^2+4)^2$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$

f.) when $x=2$, $f(x) = \dfrac{3\cdot 2}{2^2+4} = \dfrac{6}{8} = \dfrac{3}{4}$,

and when $x=-2$, $f(x) = \dfrac{3\cdot(-2)}{(-2)^2+4} = \dfrac{-6}{8} = \dfrac{-3}{4}$. Also $f(0) = \dfrac{0}{4} = 0$.

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8. *(10 points)* Give a definition of the definite integral of a function as a limit of Riemann sums. You should explain the meaning of the symbols you use in your definition.

Given an interval $[a,b]$ and a function $f(x)$ defined on $[a,b]$; for each n, split $[a,b]$ into n subintervals of length Δx (2) and for each i between 1 and n, let x_i^* be any point in the ith subinterval. (2) Then we define

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad , \quad \text{if the limit exists.}$$

(2) (2) (2) (2)

9. *(24 points)* Evaluate the indefinite integral:

[12] a. $\displaystyle\int x^2\left(\frac{x^3}{12}-7\right)^5 dx$

$$= \int x^2 (u)^5 \frac{4\,du}{x^2} = 4\int u^5\,du = 4\,\frac{u^6}{6} + C = \boxed{\frac{2}{3}\left(\frac{x^3}{12}-7\right)^6 + C}$$

$u = \frac{x^3}{12} - 7$ (2)

$du = \frac{3x^2}{12}\,dx = \frac{x^2}{4}\,dx$ (2)

$\Rightarrow dx = \frac{4\,du}{x^2}$

(2) (2) (2) (2)

[12] b. $\displaystyle\int \frac{x}{(x+2)^3}\,dx$

$$= \int \frac{u-2}{u^3}\,du = \int\left(\frac{u}{u^3} - \frac{2}{u^3}\right)du = \int\left(u^{-2} - 2u^{-3}\right)du = \frac{u^{-1}}{-1} - 2\left(\frac{u^{-2}}{-2}\right) + C =$$

$$= \frac{-1}{u} + \frac{1}{u^2} + C = \boxed{\frac{-1}{x+2} + \frac{1}{(x+2)^2} + C}$$

$u = x+2$ (2)

$du = 1\cdot dx$ (2)

$x = u-2$ (2)

(2) (2) (2)

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10. *(12 points)* Evaluate the definite integral $\int_0^1 x(x^2+1)^{1/3}\,dx$.

$u = x^2 + 1$ (2)
$du = 2x\,dx$ (2)
$x = 0 \rightarrow u = 1$ (2)
$x = 1 \rightarrow u = 2$

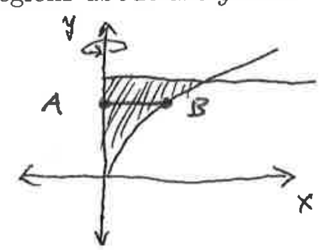
$$\int_0^1 x(x^2+1)^{1/3}\,dx = \int_1^2 x\, u^{1/3} \frac{du}{2x} = \quad (2)$$

$$= \frac{1}{2}\int_1^2 u^{1/3}\,du = \frac{1}{2}\left[\frac{u^{4/3}}{4/3}\right]_1^2 = \quad (2)$$

$$= \frac{3}{8}\left[2^{4/3} - 1^{4/3}\right] = \boxed{\frac{3}{8}\left[2^{4/3} - 1\right]} \quad (2)$$

11. *(22 points)* Find the volume of the solids obtained by revolving the shaded regions about the y-axis:

a. The region between $x = 0$, $y = 1$, and $x = y^4$.



$AB = x = y^4$

[11] By discs:

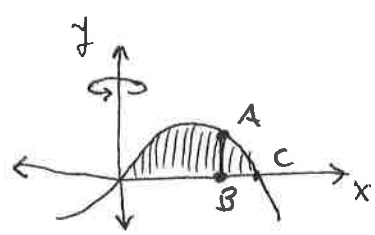
$$V = \int_0^1 \pi (AB)^2\,dy \quad (1)(1)(1)(1)$$

$$= \int_0^1 \pi (y^4)^2\,dy \quad (2)$$

$$= \pi \int_0^1 y^8\,dy = \pi\left[\frac{y^9}{9}\right]_0^1 \quad (2)(1)$$

$$= \boxed{\frac{\pi}{9}} \quad (1)$$

b. The region between $y = 0$ and $y = x - x^3$.



$AB = y = x - x^3$

$x - x^3 = 0 \leftrightarrow$
$x(1-x^2) = 0 \leftrightarrow$
$x(1-x)(1+x) = 0 \leftrightarrow$
$x = 0,\ x = 1,\ x = -1.$
So C is at $x = 1$.

By shells:

$$V = \int_0^1 2\pi x (AB)\,dx \quad (1)(1)(1)(1)$$

$$= \int_0^1 2\pi x (x - x^3)\,dx \quad (2)$$

$$= 2\pi \int_0^1 (x^2 - x^4)\,dx = 2\pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 \quad (2)(2)$$

$$= 2\pi\left[\frac{1}{3} - \frac{1}{5}\right] = 2\pi\left(\frac{5-3}{15}\right) = \boxed{\frac{4\pi}{15}} \quad (1)$$