Math 1914
Review for First Exam

The first exam is over sections 1.4, 1.5, 1.6, 1.7, 1.8, 2.1, 2.2, 2.3, and 2.4 of the text. These sections were covered on Assignments 1 through 4 and Quizzes 1 and 2.

Here is a summary of the topics covered on the test, arranged by section.

1.4. The tangent and velocity problems. This section was mostly preliminary; it's superseded by the material in section 2.1 (see below). However, you should be sure to (i) know how to find the slope of the secant line through two points on the graph of a function; (ii) know how to find the average velocity of a moving object between two times, given a formula for its position as a function of time; and (iii) know how (i) and (ii) are related. In particular, you should be familiar with diagrams such as the one in Figure 5 at the bottom of page 48 and Figures 5 and 6 on page 106. See also the two homework problems from section 1.4 which were on Assignment 1.

1.5, 1.6. The limit of a function. Calculating a limit using the limit laws. Section 1.5 just discusses the intuitive meaning of the limit of a function, without going into details about how to compute limits. For that reason I didn't assign any homework from that section. Section 1.6, on the other hand, gives lots of examples of how to calculate limits. You should also be aware of the limit laws in the red boxes on pages 62 to 64. Not that you need to consciously use these laws to do problems — rather, it's important to be aware of what you can and can't do with limits. An example of what you have to be careful about is that you can *not* say, just on the basis of the fact that $\lim_{x\to 2} f(x) = 0$, that $\lim_{x\to 2} f(x)g(x)$ is also equal to zero. Do you see why not? (Answer is below.)

1.7. The precise definition of a limit. In class we briefly went over the precise definition of limit, which is in the red box on page 73, and how to use it to prove a simple limit statement, as in Example 2 at the bottom of page 75. (See also the problem from this section on Assignment 2.) If you can do an example like that one, that's all you need to know from this section.

1.8. Continuity. You should know the definition of continuity (red box at the top of page 82). It also helps to know the fact that polynomials, quotients of polynomials, and trigonometric functions are continuous at every point in their domains. There are a few places important places in a course on calculus where the notion of continuity comes up. One of them is in the proof of the Product Rule (see below). Also, an important theorem about continuous functions is the Intermediate Value Theorem (red box on page 89); see example 9 on page 89 and the homework exercise #51 from this section.

2.1. Derivatives and rates of change. The definition of derivative is given several times in this section: once as Definition 4 in the top red box on page 107; once again on page 107 in the bottom red box, equation 5; and then twice more on page 108 in the red box numbered as equation 6. Notice that all of these definitions are saying the same thing, just with different notations. It's important to make the connections between these definitions and the pictures in Figures 6 and 8 in this section.

You might suppose that if you know the rules of differentiation, then you won't need to know these definitions, since it's easier to calculate derivatives using the rules of differentiation than using the definition. However, there are lots of ways to *use* derivatives that require you to be familiar with the definition of derivative. An example of such a use (not on this test, but coming up soon) is when you use derivatives to approximate values of a function as in Section 2.9. Another (farther along in the book) is when you use derivatives to recover a function from its integral (you can see that done on page 312).

Because understanding the definition of derivative is important, I'll ask you to be able to compute the derivative of functions using the definition of derivative, as in the problems assigned from this section, or as on Quiz 1. (If I don't specify that you have to use the definition of derivative, you can use whatever rule of differentiation you want to find the derivative.)

2.2. The derivative as a function. There were three important ideas discussed in this section. First, a derivative of a function is itself a function, and so can have its own derivative (which is called the second derivative of the original function). Second, some functions are not differentiable (do not have derivatives). Third, functions that do have derivatives are automatically continuous. Learning these simple facts will help you understand calculus better.

2.3. Differentiation formulas. These formulas are the easiest part of calculus to learn, and the easiest to remember. They are summarized in the red box at the end of the section, on page 136.

The rules that aren't completely straightforward are the product rule, quotient rule, and power rule. The product rule is the most basic of these, because you can use it (as we did in class) to derive the quotient rule and power rule. I might ask you to prove the product rule on the exam. There is a proof in the text (page 131), but if you prefer you can give the following shorter version:

Proof that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$*:* Let $y = uv$. If x increases by an amount Δx, then u will increase by an amount Δu, and v will increase by an amount Δv. The increase in y will equal the new value of y minus the old value of y, which is

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u \Delta v.$$

So

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u\Delta v + v\Delta u + \Delta u \Delta v}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right) \\ &= u\frac{dv}{dx} + v\frac{du}{dx} + 0\frac{dv}{dx} \\ &= u\frac{dv}{dx} + v\frac{du}{dx}. \end{aligned}$$

(You don't have to write out the proof exactly that way; you can put it in your own words.)

(NOTE: This is a corrected version of the proof — there was a previous version of this review sheet in which some of the terms in the expression for Δy were omitted. If you memorized the incorrect version, that's okay; I'll accept the incorrect version as correct on this exam.)

2.4. Derivatives of trigonometric functions. You should know the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, and be able to use this fact to find other limits, as in the homework problems from this section. You should also know the formulas for the derivatives of trigonometric functions (see box on page 144).

In class, we spent a fair amount of time explaining the proof that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, as well as the proofs of the formulas for the derivatives of sine and cosine, but I won't ask for these proofs on the exam.

Answer to question posed above in discussion of section 1.6: If we know that the limit of $g(x)$ exists as $x \to 2$, then according to the limit laws, we have that $\lim_{x \to 2} f(x)g(x) = 0 \cdot M$, where M stands for whatever the limit of $g(x)$ is as $x \to 2$. In this case, then, since $0 \cdot M = 0$, we'd have that $\lim_{x \to 2} f(x)g(x) = 0$. However, if the limit of $g(x)$ does *not* exist as $x \to 2$, then the limit laws do not apply to $f(x)g(x)$, and in fact the limit of $f(x)g(x)$ might not be zero. For example, if $f(x) = x - 2$ and $g(x) = 1/(x - 2)$, then $\lim_{x \to 2} f(x) = 0$, but $\lim_{x \to 2} f(x)g(x)$ is equal to 1, not 0!