EXAM 1
Math 1914
9-21-15

Name key (Green)

Row

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* Give a proof of the product rule for differentiation.

(One possible proof – an alternate one is in the textbook)

Suppose $y = uv$ and u, v are functions of x. If x increases from x to $x + \Delta x$, then u increases from u to $u + \Delta u$ and v increases from v to $v + \Delta v$, so y increases from uv to $(u+\Delta u)(v+\Delta v)$. Thus

$$\Delta y = (u+\Delta u)(v+\Delta v) - uv = \cancel{uv} + (\Delta u)v + u(\Delta v) + (\Delta u)(\Delta v) - \cancel{uv} \quad 2$$

or $\Delta y = (\Delta u)v + u(\Delta v) + (\Delta u)(\Delta v)$ 2, and $\dfrac{\Delta y}{\Delta x} = \left(\dfrac{\Delta u}{\Delta x}\right)\cdot v + u\left(\dfrac{\Delta v}{\Delta x}\right) + \Delta u\left(\dfrac{\Delta v}{\Delta x}\right)$ 2.

As $\Delta x \to 0$, then $\Delta u \to 0$ (assuming u is continuous), so

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\cdot v + u\frac{\Delta v}{\Delta x} + \Delta u\cdot\frac{\Delta v}{\Delta x}\right) = \boxed{\frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx} + 0} \quad 2$$

2. *(20 points)* Use the definition of derivative to calculate $f'(a)$, where $f(x) = \dfrac{7}{x} - 5x$.

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = \lim_{h\to 0}\left\{\frac{\left[\frac{7}{a+h} - 5(a+h)\right] - \left[\frac{7}{a} - 5a\right]}{h}\right\} =$$
2 3 2

$$= \lim_{h\to 0}\left\{\frac{\frac{7}{a+h} - \cancel{5}a - 5h - \frac{7}{a} + \cancel{5}a}{h}\right\} = \lim_{h\to 0}\left\{\frac{\frac{7}{a+h} - \frac{7}{a} - 5h}{h}\right\} =$$
3

$$= \lim_{h\to 0}\left\{\frac{\frac{7}{a+h} - \frac{7}{a}}{h} - 5\right\} = \lim_{h\to 0}\left\{\frac{\frac{7}{a+h} - \frac{7}{a}}{h}\right\} - 5 =$$
2

$$= \lim_{h\to 0}\left\{\frac{7a - 7(a+h)}{a(a+h)}\cdot\frac{1}{h}\right\} - 5 =$$
2

$$= \lim_{h\to 0}\left\{\frac{\cancel{7a} - \cancel{7a} - 7h}{a(a+h)}\cdot\frac{1}{h}\right\} - 5 = \lim_{h\to 0}\left\{\frac{-7\cancel{h}}{a(a+h)}\cdot\frac{1}{\cancel{h}}\right\} - 5 =$$
2

$$= \lim_{h\to 0}\left\{\frac{-7}{a(a+h)}\right\} - 5 = \boxed{\frac{-7}{a^2} - 5}$$
2 2

 Name ____________________

3. *(20 points)* Evaluate the limit, showing all work. (Do not use L'hopital's Rule!)

a. $\lim_{x\to -3} \dfrac{x^2+13x+30}{x+3}$ $= \lim_{x\to -3} \dfrac{\cancel{(x+3)}(x+10)}{\cancel{x+3}} = \lim_{x\to -3} (x+10) = -3+10 = \boxed{7}$

[6] 2 2 2

b. $\lim_{x\to 2} \dfrac{\frac{1}{9} - \frac{1}{(5-x)^2}}{x-2}$ $= \lim_{x\to 2} \left(\dfrac{\frac{(5-x)^2-9}{9(5-x)^2}}{(x-2)} \right)^{②} = \lim_{x\to 2} \dfrac{(5-x)^2-9}{9(5-x)^2} \cdot \dfrac{1}{x-2} =$ ②

[14]

$= \lim_{x\to 2} \dfrac{25-10x+x^2-9}{9(5-x)^2(x-2)}$ ② $= \lim_{x\to 2} \dfrac{x^2-10x+16}{9(5-x)^2(x-2)}$ ② $=$

$= \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x-8)}{9(5-x)^2\cancel{(x-2)}}$ ② $= \lim_{x\to 2} \dfrac{x-8}{9(5-x)^2}$ ② $= \dfrac{2-8}{9(5-2)^2} = \dfrac{-6}{9\cdot 9}$ ② $= \boxed{\dfrac{-2}{27}}$.

4. *(20 points)* Find the derivative. (In this and all the subsequent problems on the test, you may use any of the rules of differentiation.)

a. $y = \dfrac{5x}{4+2x^3}$

[8]

$\dfrac{dy}{dx} = \dfrac{(4+2x^3)\cdot\frac{d}{dx}(5x) - 5x\,\frac{d}{dx}(4+2x^3)}{(4+2x^3)^2}$ ④

$= \boxed{\dfrac{(4+2x^3)\cdot 5^{②} - 5x(0+6x^2)}{(4+2x^3)^2}}$ ②

b. $y = \left(5x^6 - \dfrac{1}{\sqrt{x}}\right)^2$ $= \left(5x^6 - \frac{1}{\sqrt{x}}\right)\left(5x^6 - \frac{1}{\sqrt{x}}\right)^{②} = \left(5x^6 - x^{-\frac{1}{2}}\right)\left(5x^6 - x^{-\frac{1}{2}}\right)$ ②

[12]

So by the Product Rule, ②

$\dfrac{dy}{dx} = (5x^6 - x^{-\frac{1}{2}})\dfrac{d}{dx}(5x^6 - x^{-\frac{1}{2}}) + \dfrac{d}{dx}(5x^6 - x^{-\frac{1}{2}})(5x^6 - x^{-\frac{1}{2}})$ ②

$= 2^{②}(5x^6 - x^{-\frac{1}{2}})\dfrac{d}{dx}(5x^6 - x^{-\frac{1}{2}})$

$= 2(5x^6 - x^{-\frac{1}{2}})\cdot\left[30x^5 - \left(\frac{-1}{2}\right)x^{-3/2}\right]$ ②

Name ______________________

5. *(15 points)* A point moves on a horizontal line so that its position x at time t is given by the function $x = t^3 - t^2$.

a. Find the average velocity of the point between times $t = 2$ and $t = 4$.

[5] average velocity $= \dfrac{(x \text{ at time } t=4) - (x \text{ at time } t=2)}{4-2}$ 3

$= \dfrac{(64-16)-(8-4)}{4-2} = \dfrac{48-4}{2} = \boxed{22}$ 2

b. Find the instantaneous velocity of the point at time $t = 2$.

[5] $\left.\dfrac{dx}{dt}\right|_{t=2} = \left.3t^2 - 2t\right|_{t=2} = 3\cdot 2^2 - 2\cdot 2 = \boxed{8}$

3 2

c. Find the acceleration of the point at time $t = 2$.

[5] $\left.\dfrac{d^2x}{dt^2}\right|_{t=2} = \left.6t-2\right|_{t=2} = \boxed{10}$.

3 2

6. *(10 points)* There are two points at which the graph of the equation $y = x^3 - 3x^2 - 25x$ has a tangent line with slope 20. Find the x-coordinates of these two points.

The slope of the tangent is

2 $\dfrac{dy}{dx} = 3x^2 - 6x - 25$, 2

so the slope is 20 when

$3x^2 - 6x - 25 = 20$, 2 or

$3x^2 - 6x - 45 = 0$, or

$x^2 - 2x - 15 = 0$, or 2

$(x-5)(x+3) = 0$. 2

Solving for x gives $\boxed{x = 5 \text{ or } x = -3}$.