

Instructions Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.

1. (10 points) Give a proof of the product rule for differentiation.

(One possible proof - an alternate one is in the textbook)

Suppose u and v are functions of x and $y = uv$. If x increases by an amount Δx , this will cause u to increase by an amount Δu and v by an amount Δv . The increase in y will be

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv = uv + (\Delta u)v + u(\Delta v) + (\Delta u)(\Delta v) - uv$$

$$\text{or } \Delta y = (\Delta u)v + u(\Delta v) + (\Delta u)(\Delta v), \text{ and}$$

$$\frac{\Delta y}{\Delta x} = \left(\frac{\Delta u}{\Delta x}\right)v + u\left(\frac{\Delta v}{\Delta x}\right) + \left(\frac{\Delta u}{\Delta x}\right)(\Delta v), \text{ As } \Delta x \rightarrow 0, \text{ Then } \Delta v \rightarrow 0, \text{ (assuming } v \text{ is continuous)}$$

$$\text{and } \frac{\Delta u}{\Delta x} \rightarrow \frac{du}{dx} \text{ and } \frac{\Delta v}{\Delta x} \rightarrow \frac{dv}{dx}, \text{ so } \boxed{\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \left(\frac{du}{dx}\right)v + u\left(\frac{dv}{dx}\right) + 0}$$

2. (20 points) Use the definition of derivative to calculate $f'(a)$, where $f(x) = 8x - \frac{3}{x}$.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \left\{ \frac{\left[8(a+h) - \frac{3}{a+h}\right] - \left[8a - \frac{3}{a}\right]}{h} \right\} =$$

$$= \lim_{h \rightarrow 0} \left\{ \frac{8a + 8h - \frac{3}{a+h} - 8a + \frac{3}{a}}{h} \right\} = \lim_{h \rightarrow 0} \left\{ \frac{8h + \frac{3}{a} - \frac{3}{a+h}}{h} \right\}$$

$$= \lim_{h \rightarrow 0} \left\{ 8 + \frac{\frac{3}{a} - \frac{3}{a+h}}{h} \right\} = 8 + \lim_{h \rightarrow 0} \left\{ \frac{\frac{3}{a} - \frac{3}{a+h}}{h} \right\}$$

$$= 8 + \lim_{h \rightarrow 0} \left[\frac{3(a+h) - 3a}{a(a+h)} \right] \cdot \frac{1}{h} = 8 + \lim_{h \rightarrow 0} \left[\frac{3a + 3h - 3a}{a(a+h)} \right] \cdot \frac{1}{h} =$$

$$= 8 + \lim_{h \rightarrow 0} \frac{3h}{a(a+h)} \cdot \frac{1}{h} = 8 + \lim_{h \rightarrow 0} \frac{3}{a(a+h)} = \boxed{8 + \frac{3}{a^2}}$$

3. (20 points) Evaluate the limit, showing all work. (Do not use L'Hopital's Rule!)

[6] a. $\lim_{x \rightarrow 5} \frac{x^2 - 8x + 15}{x - 5} = \lim_{x \rightarrow 5} \frac{(x-3)(x-5)}{x-5} = \lim_{x \rightarrow 5} (x-3) = 5-3 = 2$

[14] b. $\lim_{x \rightarrow -2} \frac{\frac{1}{25} - \frac{1}{(x+7)^2}}{x+2} = \lim_{x \rightarrow -2} \frac{\frac{(x+7)^2 - 25}{25(x+7)^2}}{(x+2)} = \lim_{x \rightarrow -2} \left(\frac{(x+7)^2 - 25}{25(x+7)^2} \right) \cdot \frac{1}{(x+2)}$

$= \lim_{x \rightarrow -2} \frac{x^2 + 14x + 49 - 25}{25(x+7)^2(x+2)} = \lim_{x \rightarrow -2} \frac{x^2 + 14x + 24}{25(x+7)^2(x+2)}$

$= \lim_{x \rightarrow -2} \frac{(x+2)(x+12)}{25(x+7)^2(x+2)} = \lim_{x \rightarrow -2} \frac{x+12}{25(x+7)^2} = \frac{-2+12}{25(-2+7)^2} = \frac{10}{25 \cdot 25} = \frac{2}{125}$

4. (20 points) Find the derivative. (In this and all the subsequent problems on the test, you may use any of the rules of differentiation.)

[8] a. $y = \frac{2x}{3x^3 + 7}$

$\frac{dy}{dx} = \frac{(3x^3 + 7) \frac{d}{dx}(2x) - 2x \frac{d}{dx}(3x^3 + 7)}{(3x^3 + 7)^2}$

$= \frac{(3x^3 + 7) \cdot 2 - 2x(9x^2)}{(3x^3 + 7)^2}$

[12] b. $y = \left(\frac{1}{\sqrt{x}} + 3x^4 \right)^2 = \left(\frac{1}{\sqrt{x}} + 3x^4 \right) \cdot \left(\frac{1}{\sqrt{x}} + 3x^4 \right)$, so by product rule,

$\frac{dy}{dx} = \left(\frac{1}{\sqrt{x}} + 3x^4 \right) \left[\frac{d}{dx} \left(\frac{1}{\sqrt{x}} + 3x^4 \right) \right] + \left[\frac{d}{dx} \left(\frac{1}{\sqrt{x}} + 3x^4 \right) \right] \cdot \left(\frac{1}{\sqrt{x}} + 3x^4 \right)$

$= 2 \left(\frac{1}{\sqrt{x}} + 3x^4 \right) \cdot \left(\frac{d}{dx} \left(x^{-1/2} + 3x^4 \right) \right)$ ← (It's OK to skip straight to here, using The rule $\frac{d}{dx}(u^2) = 2u \frac{du}{dx}$)

$= 2 \left(\frac{1}{\sqrt{x}} + 3x^4 \right) \cdot \left[\left(-\frac{1}{2} \right) x^{-3/2} + 12x^3 \right]$

5. (15 points) A point moves on a horizontal line so that its position x at time t is given by the function $x = t^2 + t^3$.

a. Find the average velocity of the point between times $t = 2$ and $t = 4$.

[5] average velocity = $\frac{(x \text{ at time } t=4) - (x \text{ at time } t=2)}{4-2}$ (3)

$$= \frac{(4^2 + 4^3) - (2^2 + 2^3)}{4-2} = \frac{(16+64) - (4+8)}{2} = \frac{80-12}{2} = \boxed{34}$$

b. Find the instantaneous velocity of the point at time $t = 2$.

[5] $\left. \frac{dx}{dt} \right|_{t=2} = 2t + 3t^2 \Big|_{t=2} = 2 \cdot 2 + 3 \cdot 2^2 = 4 + 12 = \boxed{16}$.

c. Find the acceleration of the point at time $t = 2$.

[5] $\left. \frac{d^2x}{dt^2} \right|_{t=2} = 2 + 6t \Big|_{t=2} = 2 + 6 \cdot 2 = \boxed{14}$.

6. (10 points) There are two points at which the graph of the equation $y = x^3 + 6x^2 + 11x$ has a tangent line with slope 2. Find the x -coordinates of these two points.

The slope is (2) $\frac{dy}{dx} = 3x^2 + 12x + 11$ (2), so the slope is 2

when $3x^2 + 12x + 11 = 2$ (2), or $3x^2 + 12x + 9 = 0$,

or $x^2 + 4x + 3 = 0$. Solving for x gives

$(x+3)(x+1) = 0$ (2), so $\boxed{x = -1 \text{ or } x = -3}$ (2)