EXAM 2
Math 1914
10-26-15

Name key (blue)

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* Find the derivative, showing all work. (You may use any of the rules of differentiation.)

[10] a. $y = \tan\left(\frac{\cos x}{x}\right)$ $= f(g(x))$, where $f(x) = \tan x$ (1), $f'(x) = \sec^2 x$ (1)

$g(x) = \frac{\cos x}{x}$ (1), $g'(x) = \frac{x(-\sin x) - \cos x \cdot 1}{x^2}$ (2),

so $f'(g(x)) = \sec^2\left(\frac{\cos x}{x}\right)$ (2), so $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$ (1) $= \boxed{\sec^2\left(\frac{\cos x}{x}\right) \cdot \left[\frac{-x\sin x - \cos x}{x^2}\right]}$ (2)

[10] b. $y = \left(\frac{1}{1+\sin(3x)}\right)^5$ (2) $= (1+\sin(3x))^{-5}$, so

$\frac{dy}{dx} = (-5)(1+\sin(3x))^{-6}$ (2) $\cdot \frac{d}{dx}(1+\sin(3x))$ (2) $= \boxed{\frac{-5}{(1+\sin(3x))^6}(0 + \cos(3x) \cdot 3)}$ (2) (2).

(You could also find $\frac{dy}{dx}$ by using

$\frac{dy}{dx} = 5\left(\frac{1}{1+\sin(3x)}\right)^4 \cdot \frac{d}{dx}\left(\frac{1}{1+\sin(3x)}\right) = $ etc...)

2. *(15 points)* Suppose y is a function of x satisfying the equation $2x+3y+y^3 = 6$. Use implicit differentiation to find:

[10] a. $\frac{dy}{dx}$ when $x = 1$ and $y = 1$.

$\frac{d}{dx}(2x + 3y + y^3) = \frac{d}{dx}(6)$ (2)

$\Rightarrow 2 + 3\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$ (1) (1) (1) (1) $\Rightarrow 2 + (3 + 3y^2)\frac{dy}{dx} = 0$ (1)

$\Rightarrow \frac{dy}{dx} = \frac{-2}{3+3y^2}$ (1); at $x=1$ and $y=1$ we have $\frac{dy}{dx} = \frac{-2}{3+3} = \boxed{\frac{-1}{3}}$ (2)

[5] b. $\frac{d^2y}{dx^2}$ when $x = 1$ and $y = 1$.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-2}{3+3y^2}\right)$ (1) $= (-2)\frac{d}{dx}\left[(3+3y^2)^{-1}\right]$ (1) $=$

$= (-2)\cdot(-1)[3+3y^2]^{-2}$ (1) $\cdot\left(0 + 6y\frac{dy}{dx}\right) = \frac{2}{(3+3y^2)^2}\cdot 6y\frac{dy}{dx}$.

So when $y = 1$ and $\frac{dy}{dx} = \frac{-1}{3}$, we have $\frac{d^2y}{dx^2} = \frac{2}{6^2}\cdot 6 \cdot \left(\frac{-1}{3}\right) = \frac{-4}{36} = \boxed{\frac{-1}{9}}$. (2)

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3. *(15 points)* Use linear approximation, and the fact that $\sqrt{2025} = 45$, to estimate $\sqrt{2015}$.

We use $f(x) \approx f(a) + f'(a)\cdot(x-a)$, (2)

with $f(x) = \sqrt{x}$, (2) $x = 2015$, and $a = 2025$. (2)

Then $f(a) = \sqrt{2025} = 45$, (1) and

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, (2) so $f'(a) = \frac{1}{2\sqrt{2025}} = \frac{1}{2\cdot 45} = \frac{1}{90}$. (1)

Then $\sqrt{2015} \approx 45 + \frac{1}{90}(2015-2025)$ (2) (3)

or $\sqrt{2015} \approx 45 + \frac{-10}{90} = \boxed{44\frac{8}{9}}$.

4. *(15 points)* Find the maximum and minimum values of $f(x) = x(2\sqrt{x} - 6)$ on the interval $[0, 100]$. Show all work.

Maximum value of $f(x)$ is 140 at $x =$ 100

Minimum value is $f(x)$ is -8 at $x =$ 4

$f(x) = 2x^{3/2} - 6x$, so $f'(x) = 2\cdot\frac{3}{2}x^{\frac{1}{2}} - 6 = 3\sqrt{x} - 6$. (2)

Critical point(s) are where $3\sqrt{x} - 6 = 0$, (2) or $\sqrt{x} = \frac{6}{3} = 2$,

or $x = 4$. (2)

Checking $f(x)$ at the endpoints $x = 0$ and $x = 100$, and at the critical points, gives

$$\begin{cases} f(0) = 0(2\cdot 0 - 6) = 0 & (2) \\ f(4) = 4(2\sqrt{4} - 6) = 4(2\cdot 2 - 6) = 4(-2) = -8 & (2) \\ f(100) = 100(2\sqrt{100} - 6) = 100(2\cdot 10 - 6) = 100\cdot 14 = 140. & (2) \end{cases}$$

So max value is 140 at $x = 100$ and min value is -8 at $x = 4$.

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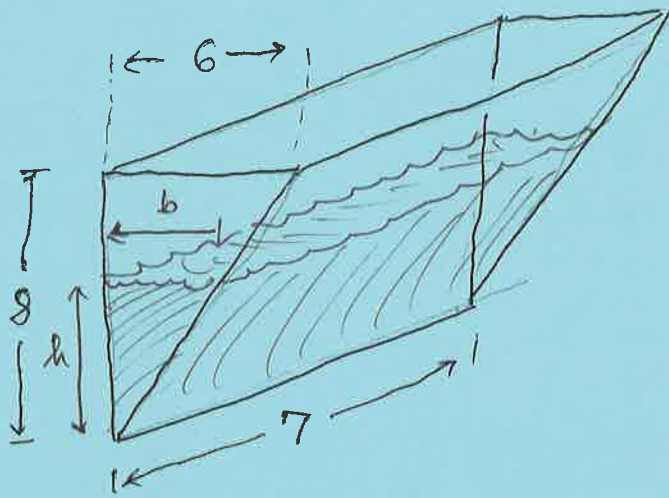
5. *(15 points)* A water trough (see diagram) has a cross-section in the shape of a triangle with base 6 feet and height 8 feet. The trough is 7 feet long. The water in the trough is rising at a rate of $\frac{1}{10}$ feet per minute. How fast is the volume of water in the trough increasing when the water is 4 feet deep?

(Note: the volume of a triangular prism is equal to the product of its length and the area of a cross-section. The area of a triangle is one-half its base times its height.)

Let h be the depth of the water and b be the distance across the trough at the water surface. Let V be the volume of water in the trough. 2



Then $V = \left(\frac{1}{2} bh\right) \cdot 7$. 2

We are given $\frac{dh}{dt} = \frac{1}{10} \frac{ft}{min}$, 2 and we want to find $\frac{dV}{dt}$ when $h = 4$.

We have $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{7}{2} bh\right) = \frac{7}{2}\left(b \frac{dh}{dt} + h \frac{db}{dt}\right)$. 2

From similar triangles, $\frac{b}{h} = \frac{6}{8} = \frac{3}{4}$, so $b = \frac{3h}{4}$. 2

So $\frac{db}{dt} = \frac{3}{4} \cdot \frac{dh}{dt} = \frac{3}{4} \cdot \frac{1}{10} = \frac{3}{40}$. 2

Also, when $h = 4$, $b = \frac{3h}{4} = 3$. 1 So

$$\frac{dV}{dt} = \frac{7}{2}\left(3 \cdot \frac{1}{10} + 4 \cdot \frac{3}{40}\right) = \frac{7}{2}\left(\frac{3}{10} + \frac{3}{10}\right) = \frac{7}{2} \cdot \frac{6}{10} = \frac{42}{20} = \boxed{\frac{21}{10}}$$

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cubic ft/min

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6. *(20 points)* For the function $f(x) = 4x^5 - 5x^4$, find the following, showing all work:

a. Critical point(s) $x = 0$ and $x = 1$

b. Interval(s) of increase $(-\infty, 0)$ and $(1, \infty)$

c. Interval(s) of decrease $(0, 1)$

d. Local maximum(s) $x = 0$

e. Local minimum(s) $x = 1$

f. Use the above information to sketch the graph.

a) $f'(x) = 20x^4 - 20x^3 = 20x^3(x-1) = 0$, so $x = 0$ or $x = 1$

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b, c)

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
sign of x^3	−	+	+
sign of $x-1$	−	−	+
sign of $f'(x)$	+	−	+
f is:	2 increasing	2 decreasing	2 increasing

d, e) local max at $x = 0$
2 local min at $x = 1$ } seen from answer to b), c)

f)

x	y
⋮	
−1	$-4-5=-9$
0	0
1	$4-5=-1$
2	$4 \cdot 32 - 5 \cdot 16 = 128 - 80 = 48$
⋮	

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