<u>2.3.5</u> Let S be a non-empty subset of $\mathbb{R}$ which is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.

<u>Proof</u>: By the Completeness Axiom, S has an infimum. Let $w = \inf S$. We want to prove that $-w = \sup T$, where $T = \{-s : s \in S\}$.

(i) (1) Let $x \in T$ (i.e. the following statements are true for all $x \in T$)

(2) Then $\exists s \in S$, $x = -s$ [def of T]

(3) $-x = s$ [(3)]

(4) (4) $-x \in S$ [(2), (3)]

(5) $-x \geq w$ [def of infimum]

(6) $x \leq -w$ [(5)]

(7) $-w$ is an upper bound of T [(1) $\Rightarrow$ (6)]

(ii) (1) Let v be an upper bound of T (The following statements are true for all upper bounds v of T)

(2) Let $s \in S$ (the following statements are true for all $s \in S$)

(3) $-s \in T$ [def of T]

(6) (4) $-s \leq v$ [(1)]

(5) $s \geq -v$ [(4)]

(6) $-v$ is a lower bound of S [(2) $\Rightarrow$ (5)]

(7) $-v \leq w$ [def of infimum]

(8) $v \geq -w$ [(7)]

(9) for every upper bound v of T, $v \geq -w$ [(1) $\Rightarrow$ (8)].

2.4.6. Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let $A+B = \{a+b : a \in A, b \in B\}$

Prove that $\sup(A+B) = \sup A + \sup B$

and $\inf(A+B) = \inf A + \inf B$.

Proof that $\sup(A+B) = \sup A + \sup B$.

Let $u = \sup A + \sup B$. We need to prove two things:

(i) u is an upper bound for $A+B$

(ii) For every upper bound v of $A+B$, $v \geq u$.

(i): (1) Let $x \in A+B$. ~~Then~~ (2) $x = a+b$ for some $a \in A$, $b \in B$. [def of $A+B$]

(4) (3) Since $a \in A$, Then $a \leq \sup A$ [def of supremum]

(4) Since $b \in B$, Then $b \leq \sup B$ [def of supremum]

(5) $a+b \leq \sup A + \sup B$ [add (3) and (4)]

(6) $a+b \leq u$ [def of u]

(7) $x \leq u$ (2)

(8) u is an upper bound of $A+B$ [since (1) $\Rightarrow$ (7)]

[Ferron + Hanzi (almost) did this. My way was less elegant, introducing extra variables, etc.]

(ii) (1) Let v be an upper bound of $A+B$

(2) Let $a \in A$.

(5) (3) Let $b \in B$

(4) $a+b \in A+B$ [def of $A+B$]

(5) $a+b \leq v$ [~~def of ...~~ (1), (4)]

(6) $b \leq v-a$ [subtract a from (5)]

(7) $v-a$ is an upper bound of B [since (3) $\Rightarrow$ (6)]

(8) $v-a \geq \sup B$ [def of supremum]

(9) $v - \sup B \geq a$ [add a to and subtract $\sup B$ from (7)]

(10) $v - \sup B$ is an upper bound of A [since (2) $\Rightarrow$ (9)]

(11) $v - \sup B \geq \sup A$ [def of supremum]

(over)

(12) $v \geq \sup A + \sup B$ [add $\sup B$ to (11)]
(13) $v \geq u$ [def of u].
(14) for every upper bound v of $A+B$, $v \geq u$ [(1) => (13)]

Proof that $\inf(A+B) = \inf A + \inf B$

(1) We could repeat the above proof with inequalities reversed and "~~sup~~" replaced by "inf" throughout. Or, we could write:

~~Let $-A = \{-a$~~

For $S \subseteq \mathbb{R}$, define $-S = \{-s : s \in S\}$, so that $s \in S$ iff $-s \in -S$. From 2.3.5, we know that if S is bdd and nonempty then $\inf S = -\sup(-S)$.

Note that $-(A+B) = (-A) + (-B)$

~~Hence~~ (because $x \in -(A+B) \iff -x \in A+B \iff (-x = a+b$ for some $a \in A, b \in B) \iff x = -a + -b \quad \exists a \in A\ \exists b \in B$

$\iff x = y + z \quad \exists y \in -A\ \exists z \in -B$

$\iff x \in -A + -B$).

So ~~by 2.3.5~~,

$$\inf A+B \underbrace{=}_{(2\text{-}3\text{-}5)} -\sup(-(A+B)) \underbrace{=}_{\text{Lemma}} -\sup((-A)+(-B)) \underbrace{=}_{\text{The part of 2.4.6 which we've already proved}} -[\sup(-A) + \sup(-B)]$$

$$= -\sup(-A) + -\sup(-B)$$

$$= \inf A + \inf B \qquad \text{[by 2.3.5)}$$