EXAM 2
Math 2423
10-22-10

Name answer key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(8 points)* Prove that the derivative of e^x is e^x.

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\left[\frac{e^{(x+h)} - e^x}{h}\right] = \lim_{h\to 0}\left[\frac{e^x\cdot e^h - e^x}{h}\right]$$

$$= \lim_{h\to 0} e^x\left[\frac{e^h-1}{h}\right] = e^x\cdot\lim_{h\to 0}\left[\frac{e^h-1}{h}\right].$$

Since $\lim_{h\to 0}\left[\frac{e^h-1}{h}\right]=1$, as follows from the definition of e, this gives

$$\frac{d}{dx}(e^x) = e^x\cdot 1 = e^x.$$

2. *(14 points)* Find the derivatives of the functions. You need not simplify your answer.

a) $y = \dfrac{e^{2x}}{e^{3x}+1}$

$$\frac{dy}{dx} = \left[\frac{(e^{3x}+1)\frac{d}{dx}(e^{2x}) - e^{2x}\frac{d}{dx}(e^{3x}+1)}{(e^{3x}+1)^2}\right]$$

$$\boxed{= \left[\frac{(e^{3x}+1)\cdot(2e^{2x}) - e^{2x}\cdot(3e^{3x})}{(e^{3x}+1)^2}\right]}$$

b) $y = e^{(xe^x)}$

$$\frac{dy}{dx} = e^{(x\cdot e^x)}\cdot\frac{d}{dx}(xe^x)$$

$$= e^{(xe^x)}\cdot\left[\frac{d}{dx}(x)\cdot e^x + x\cdot\frac{d}{dx}(e^x)\right]$$

$$= \boxed{e^{(xe^x)}\cdot\left[1\cdot e^x + x\cdot e^x\right]}$$

3. *(20 points)* Evaluate the integrals.

[10] a) $\displaystyle\int_0^2 \frac{e^{5x}}{\sqrt{e^{5x}+1}}\,dx$

$$\boxed{\begin{aligned} u &= e^{5x}+1 \\ du &= 5e^{5x}\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x=0 &\Rightarrow u = e^0+1 = 1+1 = 2 \\ x=2 &\Rightarrow u = e^{10}+1 \end{aligned}}$$

$$= \frac{1}{5}\int_{u=2}^{u=e^{10}+1}\frac{5e^{5x}}{\sqrt{u}}\,dx = \frac{1}{5}\int_2^{e^{10}+1}\frac{du}{\sqrt{u}}$$

$$= \frac{1}{5}\int_2^{e^{10}+1} u^{-1/2}\,du$$

$$= \frac{1}{5}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_2^{e^{10}+1} = \frac{2}{5}\left[\sqrt{e^{10}+1}-\sqrt{2}\right]$$

[10] b) $\int_0^{\pi/2} e^{\sin x} \cos x\,dx = \int_{u=0}^{u=1} e^u\,du = \left[e^u\right]_0^1 = e^1 - e^0 = \boxed{e-1}$ (1) (1) (1) (1) (1) (1)

$\boxed{\begin{array}{l} u = \sin x \quad (1) \\ du = \cos x\,dx \quad (1) \end{array}}$

$\boxed{\begin{array}{l} x = 0 \Rightarrow u = \sin 0 = 0 \quad (1) \\ x = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} = 1 \quad (1) \end{array}}$

4. *(14 points)* Find the intervals of increase and decrease of the function $f(x) = (3x+5)e^{-x}$.

$$f'(x) = \frac{d}{dx}(3x+5)\cdot e^{-x} + (3x+5)\frac{d}{dx}(e^{-x}) \quad (2)\ (2)$$

$$= 3\cdot e^{-x} + (3x+5)(-1)e^{-x} \quad (2) \quad = (3-3x-5)e^{-x}$$

$$= (-3x-2)e^{-x} \quad (2)$$

So $f'(x) > 0$ when $(-3x-2)e^{-x} > 0$. (1) Since $e^{-x} > 0$ for all x, this happens if and only if $-3x-2 > 0$, or when $3x+2 < 0$, (2) or when $3x < -2$, or when $x < -\frac{2}{3}$. So $\boxed{f \text{ is increasing on } (-\infty, -\tfrac{2}{3})}$

also, $f'(x) < 0 \iff (-3x-2)e^{-x} < 0$ (1) $\iff -3x-2 < 0 \iff 3x+2 > 0 \iff 3x > -2 \iff x > -\frac{2}{3}$. So $\boxed{f \text{ is decreasing on } (-\tfrac{2}{3}, \infty)}$ (2)

5. *(24 points)* Use the method of shells to find the volume obtained by rotating the given region about the specified axis. Sketch a typical shell.

a) The region between the curve $x = 2 - y - y^3$ and the lines $y = 0$ and $x = 0$; rotated around the x-axis.

[12]

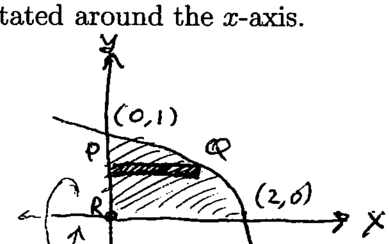
$$V = \int_0^1 2\pi x y\,dy \quad (2)$$

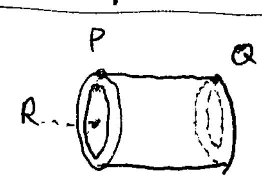
$$= \int_0^1 2\pi(2-y-y^3)y\,dy \quad (1)$$

$$= 2\pi \int_0^1 (2y - y^2 - y^4)\,dy \quad (1)$$

$$= 2\pi\left[y^2 - \frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 \quad (2)$$

$$= 2\pi\left[\left(1 - \frac{1}{3} - \frac{1}{5}\right) - 0\right] = 2\pi\left(\frac{15-5-3}{15}\right) = \frac{14\pi}{15} \quad (1)$$

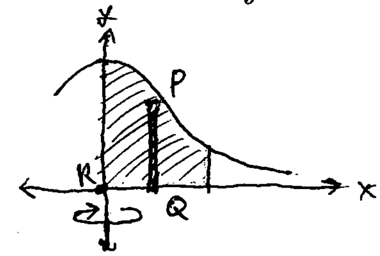




(1)

$PQ = x$ (1)
$PR = y$ (1)
Thickness $= \Delta y$ (1)
volume $= 2\pi r h \Delta y = 2\pi x y \Delta y$ (1)

[12] b) The region between the curve $y = e^{-x^2}$ and the lines $y = 0$, $x = 0$, and $x = 1$; rotated around the y-axis.



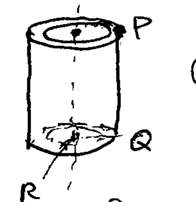
$$V = \int_0^1 2\pi y x \, dx \quad (1)$$

$$= \int_0^1 2\pi e^{-x^2} x \, dx$$

Let $u = -x^2$ (1) When $x=0$, $u=0$; when $x=1$, $u=-1$. So
$du = -2x\,dx$ (1)

$$V = \int_0^{-1} 2\pi e^u x\,dx = \pi \int_0^{-1} e^u (2x\,dx) \quad (1)$$

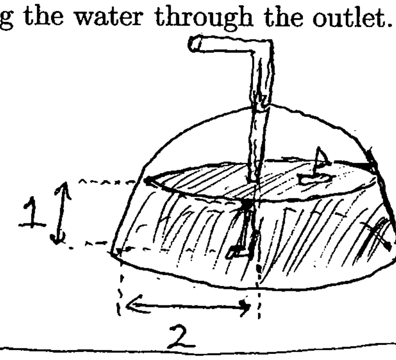
$$= \pi(-1)\int_0^{-1} e^u\,du = \pi(-1)\left[e^u\right]_0^{-1} = -\pi(e^{-1} - e^0) = \boxed{\pi[1 - 1/e]} \quad (1)$$



(1)

$PQ = y$ (1) Thickness $= \Delta x$ (1)
$QR = x$ (1)
$Vol = 2\pi h r \Delta x = 2\pi y x \Delta x$ (1)

6. *(20 points)* A hemispherical tank with radius 2 feet is filled to a depth of 1 foot with water. An outlet is located 1 foot above the top of the tank. (See diagram.) Given that water weighs 62.5 pounds per cubic foot, find the work in foot-pounds required to empty the tank by pumping the water through the outlet.



The work done to lift the layer shown in the diagram is $(3-y)\cdot(62.5)\pi(PQ)^2\Delta y$. (1)

PQ is the x-coordinate of the point Q on the circle $x^2+y^2=2^2$, so

$$PQ = x = \sqrt{2^2 - y^2} = \sqrt{4-y^2}, \quad (2)$$

and

$$(PQ)^2 = \left(\sqrt{4-y^2}\right)^2 = 4 - y^2. \quad (2)$$

So Total work is

$$W = \int_0^1 (3-y)(62.5)\pi(4-y^2)\,dy \quad (2)$$

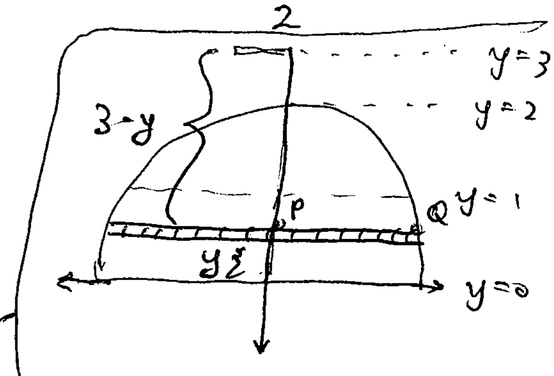
$$= (62.5)\pi\int_0^1 (12 - 4y - 3y^2 + y^3)\,dy \quad (2)$$

$$= (62.5)\pi\left[12y - 2y^2 - y^3 + \frac{y^4}{4}\right]_0^1 \quad (2)$$

$$= (62.5)\pi\left[12 - 2 - 1 + \frac{1}{4}\right] = \boxed{(62.5)\pi(9.25)} \quad (2)$$

(2) lowest layer is at $y=0$

highest layer is at $y=1$



The layer at height y above the base is a circular disk of radius PQ and thickness Δy. (1) (1) Its volume (1) is $\pi(PQ)^2\Delta y$, so its weight is

$$(62.5)\pi(PQ)^2\Delta y. \quad (1)$$

It must be lifted a distance of $(3-y)$ feet. (1)