EXAM 1
Math 2423
2-14-14

Name key

Row ____________

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

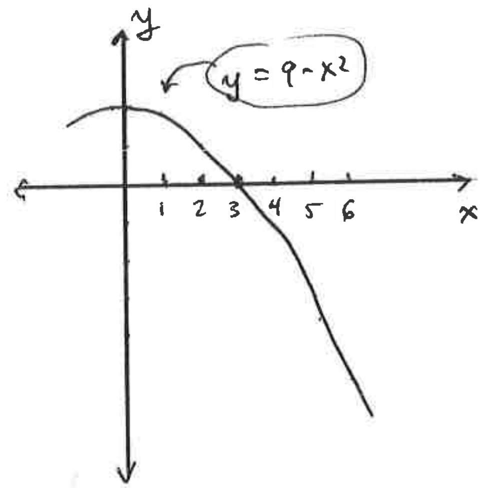
1. *(8 points)* Give the definition of the definite integral $\int_a^b f(x)\,dx$ as a limit of Riemann sums. Briefly explain the meaning of the symbols you use.

$\int_a^b f(x)\,dx$ is defined to be $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$, (1) (1) (2) (1) (1)

(2) where $[a,b]$ is split into n subintervals, each of length Δx, and x_i^* denotes a point in the i^{th} subinterval.

2. *(10 points)* Evaluate a Riemann sum for $f(x) = 9 - x^2$, $0 \le x \le 6$, with three subintervals, taking the sample points to be the midpoints of each subinterval.

$M_3 = \underline{-16}$



The subintervals are $[0,2]$, $[2,4]$, and $[4,6]$.
Their midpoints are $x_1^* = 1$, $x_2^* = 3$, $x_3^* = 5$. (2)
Their length is $\Delta x = 2$. (2)
So

$$M_3 = f(1)\cdot 2 + f(3)\cdot 2 + f(5)\cdot 2 \quad (4)$$
$$= (9-1)\cdot 2 + (9-9)\cdot 2 + (9-25)\cdot 2 \quad (2)$$
$$= 8\cdot 2 + 0 + -16\cdot 2 = 16 - 32 = \boxed{-16}$$

3. *(12 points)* Find $\dfrac{d}{dx}\displaystyle\int_0^{1/x} (\sec t)^{10}\,dt$.

Let $F(x) = \int_0^x (\sec t)^{10}\,dt$ (2) and $g(x) = 1/x = x^{-1}$.

Then $F'(x) = (\sec x)^{10}$ (by FTC part (i)) (2)
and $g'(x) = (-1)x^{-2} = -1/x^2$. (2)
So $F'(g(x)) = (\sec(\frac{1}{x}))^{10}$ (2), and the answer is

$$F'(g(x))\cdot g'(x) = (\sec(\tfrac{1}{x}))^{10}\cdot\left(\tfrac{-1}{x^2}\right).$$
(2) (2)

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4. *(36 points)* Find the indefinite integral, showing all work. Remember to express your answer as a function of x.

[12] a) $\int \frac{\sec^2(1/x)}{x^2}\,dx$ $= \int \sec^2(u)\cdot(-du) = -\int \sec^2 u\,du$

$u = 1/x = x^{-1}$ (2)

$\frac{du}{dx} = -x^{-2} = \frac{-1}{x^2}$ (2)

$-du = \frac{dx}{x^2}$

(2) (2)

$= -\tan u + C$ (2)

$= \boxed{-\tan\left(\frac{1}{x}\right) + C}$ (2)

[14] b) $\int \frac{x}{\sqrt{5x-1}}\,dx$ $= \int \frac{\left(\frac{u+1}{5}\right)}{\sqrt{u}}\cdot\frac{du}{5} = \frac{1}{25}\int \frac{u+1}{\sqrt{u}}\,du$

(1) (1) (1)

(1) $u = 5x-1$

$\frac{du}{dx} = 5$

(1) $\frac{du}{5} = dx$

Solving for x gives
$u+1 = 5x$
$\frac{u+1}{5} = x$

(2)

$= \frac{1}{25}\int (u+1)\,u^{-\frac{1}{2}}\,du$ (1)

$= \frac{1}{25}\int \left(u\cdot u^{-\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$ (2)

$= \frac{1}{25}\int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$

$= \frac{1}{25}\left(\frac{2}{3}u^{3/2} + \frac{2}{1}u^{\frac{1}{2}}\right) + C$ (2) (2)

$= \boxed{\frac{1}{25}\left(\frac{2}{3}(5x-1)^{3/2} + 2(5x-1)^{1/2}\right) + C}$

[10] c) $\int \frac{\cos x}{(\sin x+1)^3}\,dx$ $= \int \frac{du}{u^3} = \int u^{-3}\,du = \frac{u^{-2}}{-2} + C$

(2) (2) (2)

$u = \sin x + 1$ (1)

$\frac{du}{dx} = \cos x$ (1)

$du = \cos x\,dx$

$= \frac{-1}{2u^2} + C$

$= \boxed{\frac{-1}{2(\sin x+1)^2} + C}$ (2)

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5. *(12 points)* Find the value of the definite integral $\int_0^2 x^2\sqrt{1+x^3}\,dx$, showing all work.

(1) $u = 1 + x^3$

(1) $\frac{du}{dx} = 3x^2$

(1) $\frac{du}{3} = x^2\,dx$

$\rightarrow \int_0^2 x^2\sqrt{1+x^3}\,dx = \int_{u=1}^{u=9} \sqrt{u}\cdot\frac{du}{3}$ (1) (2)

$x = 0 \Rightarrow u = 1$ (1)

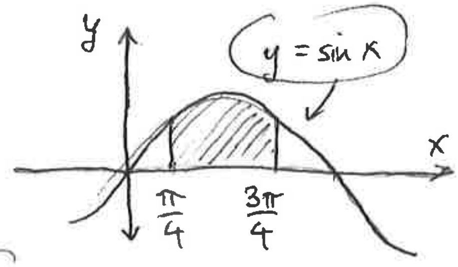
$x = 2 \Rightarrow u = 1+8 = 9$ (1)

$= \frac{1}{3}\int_1^9 u^{1/2}\,du$ (1)

$= \frac{1}{3}\cdot\frac{u^{3/2}}{3/2}\Big]_1^9$ (2)

$= \frac{2}{9}\left[u\sqrt{u}\right]_1^9 = \frac{2}{9}(9\cdot 3 - 1) = \frac{2\cdot 26}{9} = \boxed{\frac{52}{9}}$ (1)

6. *(10 points)* Find the area of the region between the the graph of $y = \sin x$, the x-axis, and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ (see diagram). Simplify your answer as much as possible.



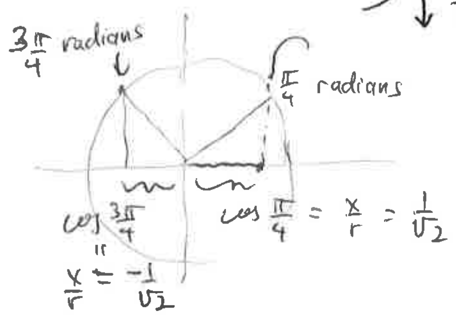
Area $= \int_{\pi/4}^{3\pi/4} \sin x\,dx$ (2)

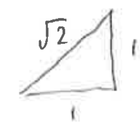
$= \left[-\cos x\right]_{\pi/4}^{3\pi/4}$ (2)

$= -\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$ (2)

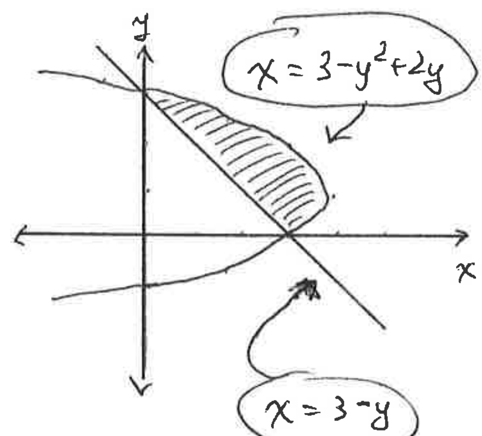
$= -\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$ (2) (2)

$= 2/\sqrt{2} = \boxed{\sqrt{2}}$





7. *(12 points)* Find the area of the region shaded in the diagram below, between the graphs of $x = 3 - y^2 + 2y$ and $x = 3 - y$.



Points of intersection: $3 - y^2 + 2y = 3 - y$

$\Rightarrow -y^2 + 2y = -y$

$\Rightarrow 3y = y^2$

$\Rightarrow 3 = y \text{ or } y = 0$ (2)

Area $= \int_0^3 \left[(3 - y^2 + 2y) - (3 - y)\right]dy$ (2) (2)

$= \int_0^3 \left[-y^2 + 3y\right]dy$

$= \left[-\frac{y^3}{3} + \frac{3y^2}{2}\right]_0^3 = \left[\left(\frac{-27}{3} + \frac{3\cdot 9}{2}\right) - 0\right] = \frac{-27}{3} + \frac{27}{2} = 27\left(-\frac{1}{3} + \frac{1}{2}\right)$ (2) (2) (2)

$= \frac{27}{6} = \boxed{9/2}$