EXAM 2
Math 2423
3-27-14

Name Key

Row

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(7 points)* Prove that the derivative of $\ln x$ is $\frac{1}{x}$.

Let $y = \ln x$.
Then $x = e^y$, so $\frac{dx}{dy} = e^y$, and therefore $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

2. *(7 points)* Find the derivative of the function $f(x) = \sqrt{1 + \ln x}$. 1

$f(x) = g(h(x))$ where $g(x) = \sqrt{x} = x^{\frac{1}{2}}$ $h(x) = 1 + \ln x$ 1

$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ 1 $h'(x) = \frac{1}{x}$ 1

2 2

so $f'(x) = g'(h(x)) \cdot h'(x) = \frac{1}{2}(1+\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x}$

3. *(16 points)* Evaluate the integral:

[8] a) $\int \frac{(\ln x)^5}{x}\, dx$ $= \int u^5\, du$ (2) $= \frac{u^6}{6} + C$ (2)

$= \frac{(\ln x)^6}{6} + C$ (2)

(1) $u = \ln x$

(1) $du = \frac{1}{x}\, dx$

[8] b) $\int \frac{e^x}{e^x + 10}\, dx$ $= \int \frac{1}{u}\, du$ (2) $= \ln u + C$ (2)

$= \ln(e^x + 10) + C$ (2)

(1) $u = e^x + 10$

(1) $du = e^x\, dx$

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4. *(16 points)*
a) Find the critical point of the function $y = xe^{-2x}$. (A critical point is a point x where $y' = 0$.)

[8]

$$y' = 1 \cdot e^{-2x} + x \cdot e^{-2x} \cdot (-2)$$

$$\Rightarrow y' = e^{-2x}(1-2x)$$

2 product rule
2 chain rule
2 factoring

Since $e^{-2x} \neq 0$, then $y' = 0$ only when $1 - 2x = 0$, or when $x = \frac{1}{2}$. 2

b) Use the second derivative test to determine whether the critical point is a local maximum or local minimum. Show all work.

[8]

$$y'' = (-2e^{-2x})(1-2x) + e^{-2x}(-2)$$

$$y'' = e^{-2x}((-2)(1-2x) + (-2))$$

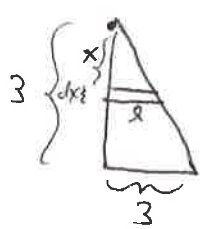
$$y'' = e^{-2x}(-4+4x). \quad 2$$

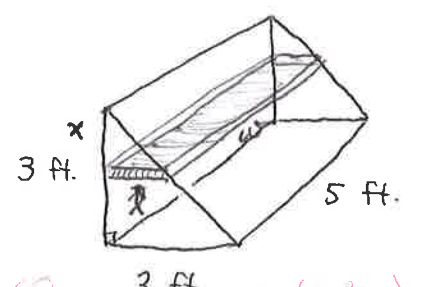
1 product rule
1 chain rule

So at $x = \frac{1}{2}$, $y'' = e^{-1}(-4+2) = (-2) \cdot e^{-1} < 0$. 2 1 So it is a local maximum 1 ☹

5. *(16 points)* The tank shown in the diagram below is full of water, weighing 62 pounds per cubic foot. Find the amount of work (in foot-pounds) required to pump all the water out of a spout at the top of the tank.

1 Let x be the distance of a layer of water from the top of the tank.
1 The thickness of the layer is dx.
Let ℓ be the length and width w the of the layer, as shown.





(2) for distance = x (or 3 − x)
(2) for $\int_0^3$
(2) for Work = Force × distance
(2) for cross-section area

By similar triangles, 1

1 $\frac{x}{\ell} = \frac{3}{3} = 1$, so $\ell = x$.

1 Also $w = 5$.

1 So volume of layer $= \ell \cdot w \cdot dx = x \cdot 5 \cdot dx$ cubic feet

1 Weight of layer $= (62 \text{ lbs/ft}^3)(5x \cdot dx) \text{ ft}^3 = 310x\, dx$ lbs.

2 Work to lift layer $=$ (weight)(distance to top of tank)

$$= x(310x\, dx)$$

$$= 310x^2\, dx$$

$$\text{Total work} = \int_{x=0}^{x=3} 310x^2\, dx = 310 \cdot \frac{x^3}{3}\Big]_0^3 = 310 \cdot \frac{27}{3} = 310 \cdot 9 = 2790 \text{ ft.-lbs.}$$

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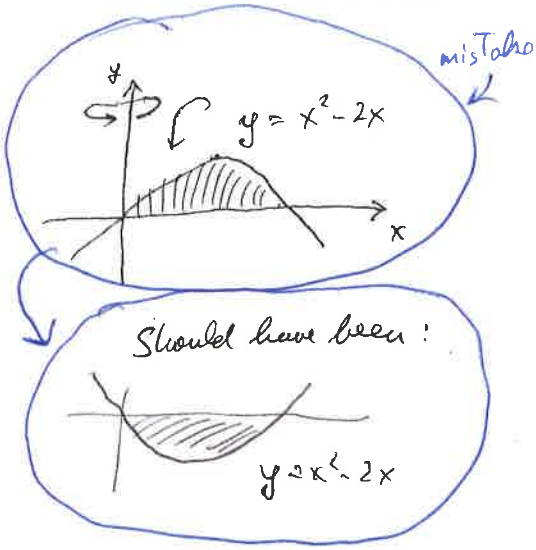
6. *(18 points)* The shaded region in the diagram lies between the curve $y = x^2 - 2x$ and the line $y = 0$. Find the volume obtained by revolving the region around the y-axis.



$$V = \int_0^2 2\pi x\left[0 - (x^2 - 2x)\right]\,dx \quad 2 \quad 2 \quad 2 \quad 2$$

$$= \int_0^2 2\pi x(-x^2 + 2x)\,dx$$

$$= 2\pi \int_0^2 \left[-x^3 + 2x^2\right] dx \quad 4$$

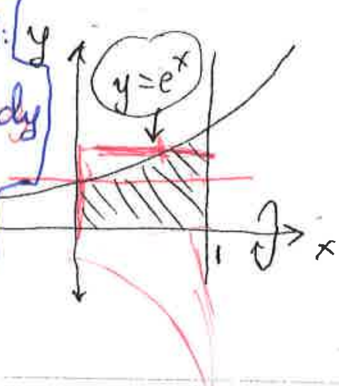
$$= 2\pi\left[\frac{-x^4}{4} + \frac{2x^3}{3}\right]_0^2 \quad 2 \quad 2$$

$$= 2\pi\left[-\frac{16}{4} + \frac{16}{3}\right] = 2\pi \cdot 16\left(-\frac{1}{4} + \frac{1}{3}\right) \quad 2$$

$$= 2\pi \cdot 16\left(\frac{1}{12}\right) = \boxed{\frac{8\pi}{3}}$$

7. *(20 points)* The shaded region in the diagram lies between the curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = 1$. Find the volume obtained by revolving the region around the x-axis.

Also valid (but harder to evaluate): $2\pi \int_1^e y(1 - \ln y)\,dy + 2\pi \int_0^1 y(1-0)\,dy$



$$V = \int_0^1 \pi y^2\,dx \quad 2 \quad 2 \quad 2$$

$$= \int_0^1 \pi (e^x)^2\,dx \quad 2$$

/10

$$= \pi \int_0^1 e^{2x}\,dx \quad 2$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2dx \end{aligned}} \quad 2 \qquad \boxed{\begin{aligned} x = 0 &\to u = 0 \\ x = 1 &\to u = 2 \end{aligned}} \quad 2$$

$$= \pi \int_0^2 e^u \frac{du}{2} \quad 2$$

$$= \frac{\pi}{2}\left[e^u\right]_0^2 = \frac{\pi}{2}\left[e^2 - e^0\right] = \frac{\pi}{2}\left[e^2 - 1\right] \quad 2 \quad 2$$