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EXAM 3
Math 2423
4-25-14

Name key

Row

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(8 points)* Prove that if $y = \arcsin x$, then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

If $y = \arcsin x$,
Then $x = \sin y$, (2) so $\dfrac{dx}{dy} = \cos y$, (2) so $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$ (2)

so $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$ (2).

2. *(16 points)* Find the limit, showing all work:

a) $\displaystyle\lim_{x\to 0} \frac{x \sin x}{1-\cos x}$

[8]

$\dfrac{0}{1-1} = \dfrac{0}{0}$

$= \displaystyle\lim_{x\to 0} \frac{x\cos x + 1\cdot \sin x}{\sin x}$ (1) (1) (1) $\dfrac{0}{0}$ $= \displaystyle\lim_{x\to 0} \frac{1\cdot \cos x + x(-\sin x) + \cos x}{\cos x}$ (1) (1) (1)

$= \dfrac{\cos 0 + 0 + \cos 0}{\cos 0}$ (1)

$= \dfrac{1+1}{1} = 2$ (1)

b) $\displaystyle\lim_{x\to 0} (1+3x)^{1/x}$ (Hint: first find the logarithm of the limit.)

[8]

$\ln\left(\displaystyle\lim_{x\to 0} (1+3x)^{1/x}\right) = \displaystyle\lim_{x\to 0} \ln\left((1+3x)^{1/x}\right) =$ (1)

$= \displaystyle\lim_{x\to 0} \frac{1}{x}\ln(1+3x) = \lim_{x\to 0} \frac{\ln(1+3x)}{x}$ (1) (1)

$= \displaystyle\lim_{x\to 0} \frac{\frac{1}{1+3x}\cdot 3}{1} = 3$, (1) (1) (1) (1) so $\displaystyle\lim_{x\to 0} (1+3x)^{1/x} = e^3$. (1)

3. *(24 points)* Find the value of the definite integral, showing all work. Simplify your answer as much as possible.

a) $\displaystyle\int_0^1 \frac{\arctan x}{1+x^2}\,dx \;=\; \int_0^{\frac{\pi}{4}} u\,du = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}}$

12]

$u = \arctan x$

$du = \frac{1}{1+x^2}\,dx$

$x = 0 \rightarrow u = \arctan 0 = 0$

$x = 1 \rightarrow u = \arctan 1 = \frac{\pi}{4}$

$= \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \boxed{\frac{\pi^2}{32}}$

b) $\displaystyle\int_{\ln(1/2)}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx \;=\; \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-u^2}}\,du = \left[\arcsin u\right]_{\frac{1}{2}}^{1}$

12]

$u = e^x$

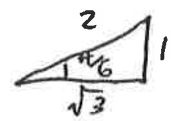
$du = e^x\,dx$

$x = \ln\frac{1}{2} \rightarrow u = e^{\ln\frac{1}{2}} = \frac{1}{2}$

$x = 0 \rightarrow u = e^0 = 1$

$= \arcsin 1 - \arcsin\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$



$= \frac{6\pi - 2\pi}{12} = \frac{4\pi}{12}$

$= \boxed{\pi/3.}$

4. *(18 points)* Evaluate the integral $\int x^2 e^{3x}\,dx$, showing all work. .

$u = x^2 \qquad dv = e^{3x}\,dx$

$du = 2x\,dx \qquad v = \frac{e^{3x}}{3}$

$\int x^2 e^{3x}\,dx = \int u\,dv = uv - \int v\,du = x^2\left(\frac{e^{3x}}{3}\right) - \int\left(\frac{e^{3x}}{3}\right)(2x)\,dx$

$\therefore \int x^2 e^{3x}\,dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int x e^{3x}\,dx$

Integrate by parts again, this time with

$u = x \qquad dv = e^{3x}\,dx$

$du = dx \qquad v = \frac{e^{3x}}{3}$

to get $\int x e^{3x}\,dx = x\cdot\left(\frac{e^{3x}}{3}\right) - \int\left(\frac{e^{3x}}{3}\right)\cdot dx$

$\therefore \int x e^{3x}\,dx = \frac{x e^{3x}}{3} - \frac{1}{3}\int e^{3x}\,dx = \frac{x e^{3x}}{3} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right) = \frac{x e^{3x}}{3} - \frac{e^{3x}}{9}.$

Putting this in the equation for $\int x^2 e^{3x}\,dx$ gives $\int x^2 e^{3x}\,dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{x e^{3x}}{3} - \frac{e^{3x}}{9}\right)$

$= \boxed{(x^2 e^{3x}/3) - (2x e^{3x})/9 + \frac{2e^{3x}}{27}}$

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5. *(16 points)* Evaluate the integral $\int \sin^3\theta \cos^6\theta \, d\theta$.

Take $u = \cos\theta$ 2

$du = -\sin\theta \, d\theta$ 2 . Then $\sin\theta \, d\theta = -du$, so

$$\int \sin^3\theta \cos^6\theta \, d\theta = \int \sin^2\theta \cos^6\theta \sin\theta \, d\theta \quad 2$$

$$= \int \sin^2\theta \, u^6 \, (-du) \quad 2$$

$$= \int (1-\cos^2\theta)\, u^6 \, (-du) \quad 2$$

$$= \int (1-u^2)\, u^6 \, (-du)$$

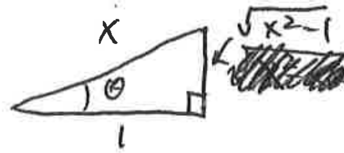
$$= \int -(u^6-u^8)\, du = \int (u^8-u^6)\, du \quad 2$$

$$= \frac{u^9}{9} - \frac{u^7}{7} + C \quad 2 \quad = \boxed{\frac{\cos^9\theta}{9} - \frac{\cos^7\theta}{7} + C} \quad 2$$

6. *(18 points)* Evaluate the integral $\int \frac{1}{x^2\sqrt{x^2-1}} \, dx$. Remember to write your answer as a function of x.

Take $x = \sec\theta$ 2

$dx = \sec\theta \tan\theta$ 2



So

$$\int \frac{1}{x^2\sqrt{x^2-1}} \, dx = \int \frac{1}{\sec^2\theta \sqrt{\sec^2\theta - 1}} \sec\theta \tan\theta \, d\theta \quad 2 \quad =$$

$$= \int \frac{1}{\sec^{\cancel{2}}\theta \, \cancel{\tan\theta}} \, \cancel{\sec\theta} \, \cancel{\tan}\,\theta \, d\theta \quad 2 \quad =$$

$$= \int \frac{1}{\sec\theta} \, d\theta \quad 2 \quad = \int \cos\theta \, d\theta \quad 2 \quad =$$

$$= \sin\theta + C \quad 2 \quad =$$

$$= \boxed{\frac{\sqrt{x^2-1}}{x} + C} \quad 2$$