EXAM 1
Math 2433
9-23-04

Name Answer key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

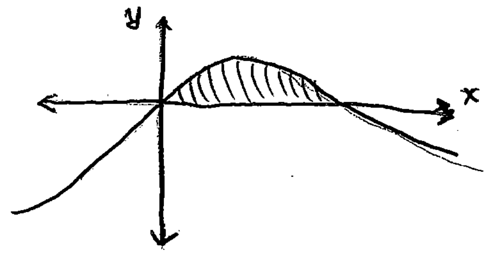
1. *(12 points)* Suppose $x = t + \sin t$, $y = 1 - \cos t$. Find

a) $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$ (2) $= \frac{\sin t}{1+\cos t}$ (2) (2)

b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ (2) $= \frac{\frac{d}{dt}\left(\frac{\sin t}{1+\cos t}\right)}{1+\cos t}$ (2) $= \frac{\left[\frac{(1+\cos t)\cos t - \sin t(-\sin t)}{(1+\cos t)^2}\right]}{(1+\cos t)}$ (2)

$= \frac{\cos t + \cos^2 t + \sin^2 t}{(1+\cos t)^3} = \frac{\cos t + 1}{(1+\cos t)^3} = \frac{1}{(1+\cos t)^2}.$

2. *(18 points)* The diagram shows the curve given by $x = t^3 + t$, $y = t - t^2$. The shaded region lies between the curve and the x-axis. Find its area.



$y = 0$ when $t - t^2 = 0$
$t(1-t) = 0$
$t = 0$ or $t = 1$. (2)

When $t=0$, $x=0$; and when $t=1$, $x=2$. (2)

$$\text{Area} = \int_{x=0}^{x=2} y\,dx \;(2) = \int_{t=0}^{t=1} (t-t^2)(3t^2+1)\,dt \quad (2)(2)(2)$$

(since $dx = (3t^2+1)dt$)

$$= \int_0^1 (3t^3 - 3t^4 + t - t^2)\,dt \quad (2)$$

$$= \left[\frac{3t^4}{4} - \frac{3t^5}{5} + \frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 \quad (2)$$

$$= \left(\frac{3}{4} - \frac{3}{5} + \frac{1}{2} - \frac{1}{3}\right) - (0) \;(2) = \frac{45 - 36 + 30 - 20}{60} = \frac{19}{60}$$

Name ______________________

3. *(10 points)* Use an integral to find the length of the curve given by $x = 1 + 2\sin t$, $y = 3 + 2\cos t$, for $0 \le t \le \frac{\pi}{10}$.

$$\text{length} = \int_0^{\pi/10} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/10} \sqrt{(2\cos t)^2 + (-2\sin t)^2}\, dt =$$ (2) (2)

$$= \int_0^{\pi/10} \sqrt{4\cos^2 t + 4\sin^2 t}\, dt = \int_0^{\pi/10} \sqrt{4}\, dt = \int_0^{\pi/10} 2\, dt =$$ (2)

$$= \Big[2t\Big]_{t=0}^{t=\pi/10} = \frac{2\pi}{10} = \boxed{\frac{\pi}{5}}.$$ (2) (2)

4. *(10 points)* A curve is given in polar coordinates by the equation $r = 5\cos\theta$.

a) Find a Cartesian equation for the curve.

$$r = 5\cos\theta \Rightarrow r^2 = 5r\cos\theta \Rightarrow \boxed{x^2 + y^2 = 5x}$$

(2) (1) (2)

b) Complete the square in the Cartesian equation to put it in the form of the equation of a circle. Give the center and radius of the circle, and sketch the circle.

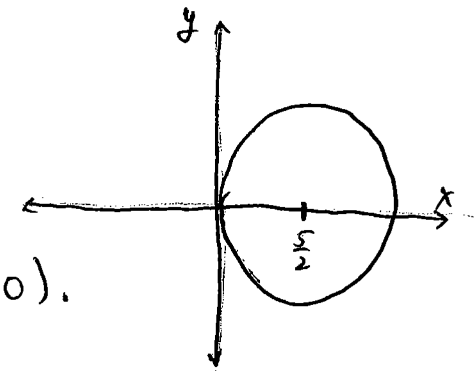
$$x^2 - 5x + y^2 = 0$$

$$x^2 - 2\cdot\frac{5}{2}x + \frac{25}{4} + y^2 = \frac{25}{4}$$ (2)

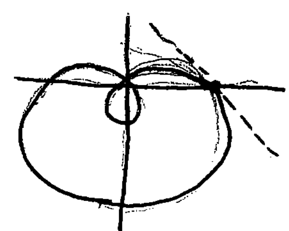
$$\boxed{\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4} = \left(\frac{5}{2}\right)^2}$$ (1)

(2)

This is a circle with radius $\frac{5}{2}$ and center $\left(\frac{5}{2}, 0\right)$.



5. *(14 points)* Find the slope of the tangent line to the polar curve $r = 2 - 3\sin\theta$ at the point where $\theta = 0$ (see diagram).



$$y = r\sin\theta = 2\sin\theta - 3\sin^2\theta$$ (2) (2)

$$x = r\cos\theta = 2\cos\theta - 3\sin\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{2\cos\theta - 6\sin\theta\cos\theta}{-2\sin\theta - (3\cos^2\theta + (\sin\theta)(-\sin\theta))}$$ (2) (2) (2)

when $\theta = 0$,

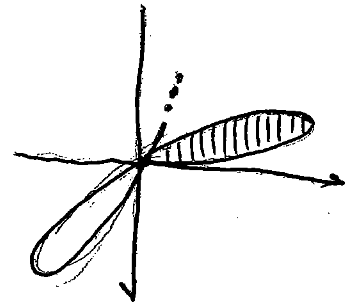
$$\frac{dy}{dx} = \frac{2\cdot\cos 0 - 6\sin 0\cos 0}{-2\cdot\sin 0 - 3\cos^2 0 + 3\sin^2 0} = \frac{2 - 0}{0 - 3 + 0} = \boxed{-\frac{2}{3}}.$$ (2) (2)

Name ______________________

6. *(18 points)* Find the area of one loop of the curve $r = \sin 6\theta$ (see diagram).

The loop begins at the origin at $\theta = 0$. The tip of the loop is at $\theta = \frac{\pi}{12}$ (15°) and $r = 1$; and the loop returns to the origin at $\theta = \frac{\pi}{6}$ (30°). So one loop is covered by the range $0 \le \theta \le \frac{\pi}{6}$. (2)

θ	6θ	$r = \sin 6\theta$
0	0	0
$\frac{\pi}{12}$	$\frac{\pi}{2}$	1
$\frac{\pi}{6}$	π	0
$\vdots$	$\vdots$	$\vdots$



The area is $\frac{1}{2}\int_0^{\pi/6} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/6} \sin^2 6\theta\,d\theta = \frac{1}{12}\int_0^{\pi} \sin^2 u\,du =$ (2) (2) (2) (2) (2)

$\boxed{\begin{array}{l} u = 6\theta \\ du = 6\,d\theta \end{array}}$ (2)

$$= \frac{1}{12}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\pi} = \frac{1}{12}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right] = \boxed{\frac{\pi}{24}.}$$

(2) (2)

7. *(18 points)* Find the area inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$.

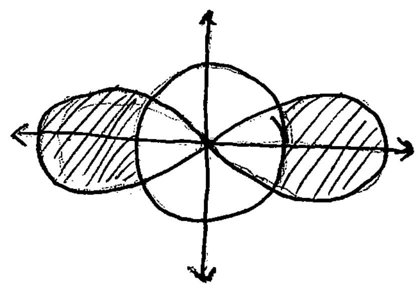
The curves $r^2 = 6\cos 2\theta$ and $r = \sqrt{3}$ (or $r^2 = 3$) intersect when

$$6\cos 2\theta = 3, \quad (2)$$

or $\cos 2\theta = \frac{1}{2}$ (2)

$$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$$



The intersection in the first quadrant is at $\theta = \frac{\pi}{6}$. The total area will be 4 times the area in the first quadrant, which is

$$(2)\ 4\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}(6\cos 2\theta)\,d\theta - \int_0^{\frac{\pi}{6}} \frac{1}{2}\cdot 3\,d\theta\right] = \quad (2)\ (2)$$

$$= \int_0^{\frac{\pi}{6}} 12\cos 2\theta\,d\theta - \int_0^{\frac{\pi}{6}} 6\,d\theta = \left[12\,\frac{\sin 2\theta}{2}\right]_0^{\pi/6} - \left[6\theta\right]_0^{\pi/6} \quad (2)\ (2)$$

$\begin{array}{l} u = 2\theta \\ du = 2\,d\theta \end{array}$ (2)

$$= \left(12\,\frac{\sin(\pi/3)}{2} - 0\right) - \left(6\cdot\frac{\pi}{6} - 0\right) \quad (2)$$

$$= \boxed{3\sqrt{3} - \pi}$$