EXAM 2
Math 2433
10-28-04

Name Answer key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(8 points)* Find the sum of the series $9 - 8 + \frac{64}{9} - \frac{512}{81} + \ldots$.

(2)

Use $\frac{a}{1-r} = a + ar + ar^2 + \cdots$ with $a = 9$ (2) and $r = \frac{-8}{9}$ (2).

The sum is $\frac{9}{1-(-\frac{8}{9})} = \frac{9}{(17/9)} = \frac{81}{17}$ (2)

2. *(20 points)* Determine (with explanation) whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ [10]

Here $a_n = \frac{100^n}{n!}$ and $a_{n+1} = \frac{100^{n+1}}{(n+1)!}$, so

(2) (2)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100^{n+1}}{100^n} \cdot \frac{n!}{(n+1)!}$$

(2) (2)

$$= \lim_{n\to\infty} 100 \cdot \frac{1}{n+1} = 0.$$

Since $0 < 1$ (2), the series converges by the Ratio Test.

b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

~~[illegible]~~

(1)

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\,dx$$

(2) $\boxed{\begin{matrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{matrix}}$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{\sqrt{u}} \quad (1)$$

(2) (2)

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-1/2}\,du = \lim_{b\to\infty} \left[2u^{1/2}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left[2\sqrt{\ln b} - 2\sqrt{\ln 2}\right] = \infty$$

So the series diverges by the Integral Test. (2)

3. *(20 points)* Determine (with explanation) whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n^3}$ is absolutely convergent, conditionally convergent, or divergent.

You can use The Alternating Series test*: since $\lim_{x\to\infty} \frac{\ln x}{x^3} = \lim_{x\to\infty} \frac{1/x}{3x^2}$ ② (by L'hopital's Rule) $= \lim_{x\to\infty} \frac{1}{3x^3} = 0$ ②, Then by The Alt. Series Test ② The series converges. ②

For $\sum \frac{\ln n}{n^3}$ we can use The limit comparison test, comparing to $\sum \frac{1}{n^2}$. ②

Since $\lim_{n\to\infty} \frac{\left(\frac{\ln n}{n^3}\right)}{\left(\frac{1}{n^2}\right)}$ ② $= \lim_{n\to\infty} \frac{n^2 \ln n}{n^3}$ ② $= \lim_{n\to\infty} \frac{\ln n}{n}$ ② $= \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1}$ ① $= 0$, ②

Then by The limit comparison test (as discussed in class, which goes beyond what is in the text), since $\sum \frac{1}{n^2}$ ② converges then $\sum \frac{\ln n}{n^3}$ ① converges. Hence ④ $\boxed{\text{ABSOLUTE CONVERGENCE}}$

4. *(20 points)* Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n5^n}$.

Here $a_n = \frac{x^n}{n5^n}$ and $a_{n+1} = \frac{x^{n+1}}{(n+1)5^{n+1}}$ ②, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| ② = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{x^n}\right| = \lim_{n\to\infty} \frac{n}{n+1} \frac{5^n}{5^{n+1}} \frac{|x|^{n+1}}{|x|^n} =$$

$$= \lim_{n\to\infty} \frac{n}{n+1} ② \cdot \frac{|x|}{5} = \frac{|x|}{5} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} ② = \frac{|x|}{5}\left(\frac{1}{1+0}\right) = \frac{|x|}{5}. ②$$

So, by the Ratio Test, the series converges when $\frac{|x|}{5} < 1$, or $|x| < 5$, ② and diverges when $|x| > 5$. We still have to check when $|x| = 5$:

When $x = 5$, the series is ① $\sum_{n=1}^{\infty} \frac{5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges ② (harmonic series is divergent)

When $x = -5$, the series is ① $\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges ② (by Alt. Series Test)

So the series converges for $-5 \le x < 5$. ② The radius of convergence is 5.

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* This isn't necessary though: since we show in the next paragraph that $\sum \frac{\ln n}{n^3}$ converges, it follows that $\sum \frac{(-1)^n \ln n}{n^3}$ converges absolutely.

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5. *(20 points)* Find the Maclaurin series for $f(x) = (1+x)^{-2}$, showing all work. You do not have to give a formula for the terms in the series, but you should give at least the first four terms of the series.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = \frac{f^{(n)}(0)}{n!}$
0	$(1+x)^{-2}$	1 (2)	$\frac{1}{0!} = 1$
1	$-2(1+x)^{-3}$ (2)	-2 (2)	$\frac{-2}{1!} = -2$
2	$(-3)(-2)(1+x)^{-4}$ (2)	$3 \cdot 2$ (3)	$\frac{3 \cdot 2}{2!} = 3$
3	$(-4)(-3)(-2)(1+x)^{-5}$ (2)	$-4 \cdot 3 \cdot 2$	$\frac{-4 \cdot 3 \cdot 2}{3!} = -4$
$\vdots$			

(3)

The Maclaurin series is (2)

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

or (2)

$$(1+x)^{-2} = 1 + (-2)x + 3x^2 + (-4)x^3 + \dots$$

$$\left(= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \right)$$

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6. *(12 points)* Suppose $f(x) = \dfrac{x}{1+x^3}$.

a) Find a power series representation for $f(x)$. You do not have to give a formula for the terms in the series, but you should give at least the first four terms of the series, not including zero terms. (Note: for this function, using the formula for the Maclaurin series would be too complicated; use another method instead.)

Use $\dfrac{a}{1-r} = a + ar + ar^2 + ar^3 + \dots$ 2.

with $a = x$ 2 and $r = (-x^3)$ 2. Then we get

$$\frac{x}{1+x^3} = \frac{x}{1-(-x^3)} = x + x(-x^3) + x(-x^3)^2 + x(-x^3)^3 + \dots \quad 2$$

$$= x - x^4 + x^7 - x^{10} + \dots$$

b) Use your answer to part a) to find a series whose sum is $\int_0^1 f(x)\,dx$. (Give at least the first four terms of the series.)

$$\int_0^1 \frac{x}{1+x^3}\,dx = \int_0^1 (x - x^4 + x^7 - x^{10} + \dots)\,dx$$

2

$$= \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} + \dots \right]_0^1$$

$$= \left[\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \dots \right] \quad 2$$