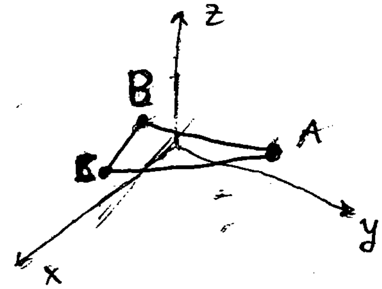
EXAM 3
Math 2433
12-3-04

Name ______________________

**Instructions** *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(16 points)* In the diagram, the vertices of the triangle $ABC$ are at $A(0,5,1)$, $B(6,3,5)$, and $C(6,2,4)$.




a) Find the cosine of the angle at $A$.

[10]

$\vec{AB} = \langle 6-0, 3-5, 5-1\rangle = \langle 6,-2,4\rangle$ 2

$\vec{AC} = \langle 6-0, 2-5, 4-1\rangle = \langle 6,-3,3\rangle$ 2

$\cos\theta = \dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \dfrac{6\cdot 6+(-2)(-3)+4\cdot 3}{\sqrt{6^2+(-2)^2+4^2}\,\sqrt{6^2+(-3)^2+3^2}}$ 2 2 2

$= \dfrac{54}{\sqrt{56}\sqrt{54}} = \sqrt{\dfrac{54}{56}} = \sqrt{\dfrac{27}{28}}$

b) Show that the angle at $C$ is a right angle.

[6]

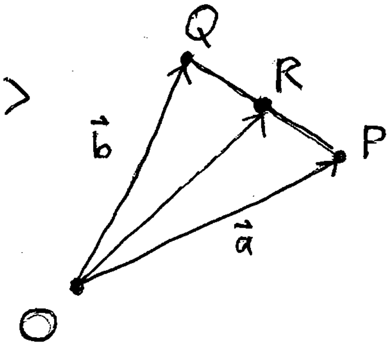
$\vec{CA} = \langle -6,3,-3\rangle$

$\vec{CB} = \langle 0, 1, 1\rangle$ 2

, so $\vec{CA}\cdot\vec{CB} = -6\cdot 0 + 3\cdot 1 + (-3)\cdot 1 = 3-3=0$ 2

Since $\vec{CA}\cdot\vec{CB}=0$, Then $\vec{CA}$ is $\perp$ to $\vec{CB}$. 2

2. *(10 points)* In the diagram, $\mathbf{a} = \overrightarrow{OP} = \langle 7,3\rangle$ and $\mathbf{b} = \overrightarrow{OQ} = \langle 1,5\rangle$. The point $R$ is halfway along the line segment from $P$ to $Q$. Find




a) $\overrightarrow{PQ} =$ $\langle -6,2\rangle$ 3

$\vec{PQ} = \vec{b}-\vec{a} = \langle 1,5\rangle - \langle 7,3\rangle$
$= \langle -6,2\rangle$

b) $\overrightarrow{PR} =$ $\langle -3,1\rangle$ 4

$\vec{PR} = \frac{1}{2}\cdot\vec{PQ} = \langle -3,1\rangle$

c) $\overrightarrow{OR} =$ $\langle 4,4\rangle$ 3

$\vec{OR} = \vec{OP} + \vec{PR}$
$= \langle 7,3\rangle + \langle -3,1\rangle = \langle 4,4\rangle$

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3. *(12 points)* A sphere has center at $C(0,3,4)$ and radius 5.

a) Write an equation of the sphere. (4) (2)

[6] $(x-0)^2 + (y-3)^2 + (z-4)^2 = 5^2$

b) Is the point $P(4,1,8)$ inside the sphere or outside the sphere? Explain your answer.

[6] The point P is inside the sphere if the distance from P to C is less than or equal to the radius of the sphere; otherwise P is outside the sphere. (3)

Since $\|\vec{PC}\| = \sqrt{\langle 4-0, 1-3, 8-4\rangle} = \sqrt{\langle 4,-2,4\rangle} = \sqrt{16+4+16} = 6$, (3) and 6 is greater than 5, the radius of the sphere, then P is <u>outside</u> the sphere.

4. *(10 points)* A line passes through the point $P(-4,6,1)$ and is parallel to the vector $\langle 2,-2,1\rangle$. Find the coordinates of the point where this line intersects the $yz$-plane.

The line has parametric equations $\begin{cases} x = -4+2t \\ y = 6-2t \\ z = 1+t \end{cases}$. (5)

The line intersects the yz-plane where $x=0$, (2) or $-4+2t=0$, or $t=2$.

When $t=2$ then $\begin{cases} x=0 \\ y=2 \\ z=3 \end{cases}$ (3) so the point of intersection is $(0,2,3)$.

5. *(22 points)* Find the equation of the plane passing through the points $A(-2,1,1)$, $B(4,5,3)$, and $C(2,1,2)$.

$\vec{AB} = \langle 4+2, 5-1, 3-1\rangle = \langle 6,4,2\rangle$ (4)

$\vec{AC} = \langle 2+2, 1-1, 2-1\rangle = \langle 4,0,1\rangle$

A normal vector to the plane is (7)

$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 4 & 2 \\ 4 & 0 & 1 \end{vmatrix} = \vec{i}(4-0) - \vec{j}(6-8) + \vec{k}(0-16)$ (2)

$= 4\vec{i} + 2\vec{j} - 16\vec{k}$

or $\vec{n} = \langle 4,2,-16\rangle$. Therefore an equation of the plane is (6)

$4(x+2) + 2(y-1) + (-16)(z-1) = 0$ (3)

or $4x+2y-16z = -22$.

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6. *(10 points)* Find the equation of the plane passing through $P(4, 1, 6)$ and perpendicular to the line

$$\frac{x-2}{4} = \frac{y-1}{7} = \frac{z}{-4}.$$

3

The line is parallel to $\langle 4, 7, -4 \rangle$, so $\langle 4, 7, -4 \rangle$ 2 is normal to the plane. So an equation of the plane is

3

$$4(x-4) + 7(y-1) + (-4)(z-6) = 0$$

2

or $4x + 7y - 4z = -1$

7. *(20 points)* The planes $-2x - y + z = 12$ and $-x + 3y - z = 6$ intersect in a line $L$. Find parametric equations for $L$. First we look for a point on the line.

Putting $x = 0$ we get $-y + z = 12$ and $3y - z = 6$.

So $3y - (12 + y) = 6$, or $2y = 18$, $y = 9$. Thus $z = 21$. Therefore $(x, y, z) = (0, 9, 21)$ is a point on $L$. 5

Next, to find another point put, say, $y = 0$. This gives

$-2x + z = 12$ and $-x - z = 6$, so $-2(-6 - z) + z = 12$ or $3z = 0$, $z = 0$. Therefore $x = -6$. So $(x, y, z) = (-6, 0, 0)$ is a point on $L$. 5

Since $P(0, 9, 21)$ and $Q(-6, 0, 0)$ are on $L$, then $\vec{PQ} = \langle -6, -9, -21 \rangle$ 3 is parallel to $L$. So param. equs for $L$ are

3

$$\begin{cases} x = 0 - 6t & 4 \\ y = 9 - 9t \\ z = 21 - 21t \end{cases}$$

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Alternate solution: Since $L$ is in the plane $-2x - y + z = 12$, then $L$ is perpendicular to $\langle -2, -1, 1 \rangle$. 2 Also since $L$ is in the plane $-x + 3y - z = 6$, then $L$ is perpendicular to $\langle -1, 3, -1 \rangle$. 2

So $L$ is parallel to $\langle -2, -1, 1 \rangle \times \langle -1, 3, -1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & 1 \\ -1 & 3 & -1 \end{vmatrix} =$ 4

$= \vec{i}(1 - 3) - \vec{j}(2 + 1) + \vec{k}(-6 - 1) = \langle -2, -3, -7 \rangle$. So param equs. for $L$ are

3

$$\begin{cases} x = 0 - 2t \\ y = 9 - 3t & 4 \\ z = 21 - 7t \end{cases}$$

(using the fact, done above, that $P(0, 9, 21)$ is on $L$.) 5