FINAL EXAM Name *Answer key*
Math 2433
12-15-04

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(18 points)* Suppose $x = t^3 + t$, $y = \sin(t^2)$. Find

a) $\frac{dy}{dx}$ $= \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$ (3) $= \dfrac{\cos(t^2)\cdot 2t \ (4)}{3t^2+1 \ (2)}$

[9]

b) $\frac{d^2y}{dx^2}$ $= \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ (3) $= \dfrac{\frac{d}{dt}\left(\frac{\cos(t^2)\cdot 2t}{3t^2+1}\right)}{(3t^2+1)}$ (2) $= \dfrac{\left[\dfrac{\{(3t^2+1)[-\sin(t^2)\cdot 2t\cdot 2t + \cos(t^2)\cdot 2] - \cos(t^2)\cdot 2t(3t^2+1)'\}}{(3t^2+1)^2}\right]}{(3t^2+1)}$ (4)

[9]

2. *(18 points)* A curve is given by the vector equation $\mathbf{r} = \langle 5t, 3\sin t, 6t^2 + 4t\rangle$.

a) Find a vector which is tangent to the curve at the origin $(0, 0, 0)$. (2)

[6] $\vec{r}'(t) = \langle 5, 3\cos t, 12t+4\rangle$ (2). At $(0,0,0)$ we have $t = 0$,

(2)

so (tangent vector at $(0,0,0)$) $= \vec{r}'(0) = \langle 5, 3\cos 0, 12\cdot 0+4\rangle = \langle 5,3,4\rangle$

b) Find a parametric equation for the tangent line to the curve at the origin.

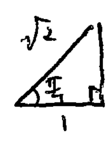
[6] Line thru $(0,0,0)$ parallel to $\langle 5,3,4\rangle$ has parametric equations (2)

$$\begin{cases} x = 0 + 5t \\ y = 0 + 3t \\ z = 0 + 4t \end{cases}$$ (4)

c) Find the angle between the curve and the x-axis at the origin. (2)

[6] angle between $\langle 5,3,4\rangle$ and $\langle 1,0,0\rangle$ is θ, where

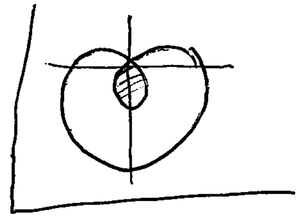
$$\cos\theta = \frac{\langle 5,3,4\rangle\cdot\langle 1,0,0\rangle}{|\langle 5,3,4\rangle|\cdot|\langle 1,0,0\rangle|} \ (2) = \frac{5\cdot 1+3\cdot 0+4\cdot 0}{\sqrt{5^2+3^2+4^2}\,\sqrt{1^2}} = \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}}$$

So $\cos\theta = \frac{1}{\sqrt{2}}$ (1), so $\theta = \frac{\pi}{4}$ (1) ()

3. *(24 points)* Find the area inside the small loop of the curve $r = 1 - 2\sin\theta$ (shaded area in the diagram).

$r = 1 - 2\sin\theta = 0$ when $\sin\theta = \frac{1}{2}$, or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

So the curve passes thru the origin at angles of $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



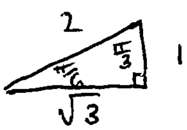
Half of the loop lies between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$, so

$$\text{Area} = 2 \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - 2\sin\theta)^2 \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[1 + 4\sin^2\theta - 4\sin\theta\right] d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[1 + 2 - 2\cos 2\theta - 4\sin\theta\right] d\theta = \left[\theta + 2\theta - \sin 2\theta + 4\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[3\theta - \sin 2\theta + 4\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 3\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - \sin 2\cdot\frac{\pi}{2} + \sin 2\cdot\frac{\pi}{6} + 4\cos\frac{\pi}{2} - 4\cos\frac{\pi}{6}$$

$$= 3\cdot\frac{\pi}{3} - \sin\pi + \sin\frac{\pi}{3} + 4\cos\frac{\pi}{2} - 4\cos\frac{\pi}{6}$$



$$= \pi - 0 + \frac{\sqrt{3}}{2} + 0 - 4\cdot\frac{\sqrt{3}}{2} = \boxed{\pi - \frac{3\sqrt{3}}{2}}.$$

4. *(14 points)* Find the length of the curve given in polar coordinates by $r = e^{3\theta}$ for $0 \le \theta \le 1$.

$$L = \int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_0^1 \sqrt{(e^{3\theta})^2 + (3e^{3\theta})^2} \, d\theta$$

$$= \int_0^1 \sqrt{e^{6\theta} + 9e^{6\theta}} \, d\theta = \int_0^1 \sqrt{10 e^{6\theta}} \, d\theta = \sqrt{10} \int_0^1 \sqrt{e^{6\theta}} \, d\theta =$$

$$= \sqrt{10} \int_0^1 e^{3\theta} \, d\theta = \frac{\sqrt{10}}{3} e^{3\theta} \Big]_{\theta=0}^{\theta=1} = \frac{\sqrt{10}}{3}\left[e^3 - e^0\right]$$

$$= \boxed{\frac{\sqrt{10}}{3}\left[e^3 - 1\right]}.$$

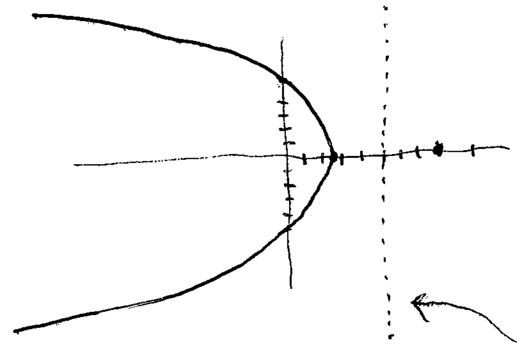
 Name ____________________

5. *(12 points)* A curve is given in polar coordinates by $r = \dfrac{5}{1+\cos\theta}$. Sketch the curve, identify which kind of conic it is, and give the equation of the directrix.

Symmetric about x-axis ($\theta \to -\theta$)

θ	r
0	$5/2$
$\frac{\pi}{3}$	$10/3$
$\frac{\pi}{2}$	5
$\frac{2\pi}{3}$	10
π	undefined

8



It is a parabola. 2

For $r = \dfrac{ed}{1+e\cos\theta}$, the directrix is at $x = d$, so since $e = 1$ and $ed = 5$ in this case, we have $d = 5$. The directrix is the line $x = 5$. 2

6. *(12 points)* Consider the surface in xyz-space defined by the equation $9x^2 - 4y^2 - 36z^2 + 36 = 0$.

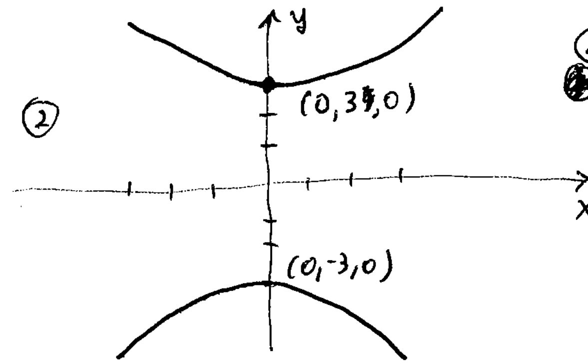
a) Sketch the trace of the surface in the xy-plane, and find the vertices.

[6] $9x^2 - 4y^2 + 36 = 0$ 2

$\Rightarrow \dfrac{x^2}{4} - \dfrac{y^2}{9} + 1 = 0$

$\Rightarrow \dfrac{y^2}{3^2} - \dfrac{x^2}{2^2} = 1$

2



2 The vertices are at $(0,3,0)$ and $(0,-3,0)$.

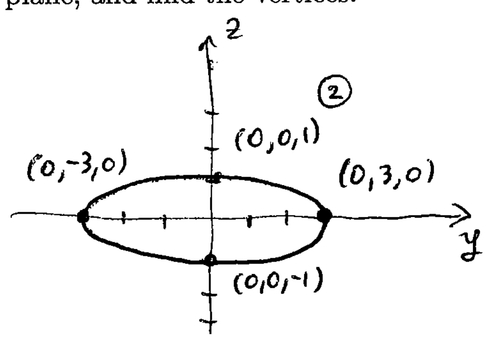
b) Sketch the trace of the surface in the yz-plane, and find the vertices.

[6] $-4y^2 - 36z^2 + 36 = 0$ 2

$\dfrac{y^2}{9} + z^2 - 1 = 0$

$\dfrac{y^2}{3^2} + z^2 = 1$

2



2 The vertices are at $(0,\pm 3,0)$ and $(0,0,\pm 1)$.

7. *(18 points)* Line L_1 is given by the vector equation $\mathbf{r} = \langle 1,-6,2\rangle + t\langle 1,2,1\rangle$, and line L_2 is given by the vector equation $\mathbf{r} = \langle 0,4,1\rangle + t\langle 2,1,2\rangle$. The lines intersect at the point $(8,8,9)$. Find an equation for the plane containing lines L_1 and L_2.

4

The plane is parallel to $\langle 1,2,1\rangle$ and $\langle 2,1,2\rangle$, so it is normal to $\vec{n} = \langle 1,2,1\rangle \times \langle 2,1,2\rangle$ 2 $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$ 2 $= \vec{i}(2\cdot 2 - 1\cdot 1) - \vec{j}(1\cdot 2 - 2\cdot 1) + \vec{k}(1\cdot 1 - 2\cdot 2)$

4

$= 3\vec{i} + 0\vec{j} - 3\vec{k}$.

6

So an equation for the plane is $3(x-8) + 0(y-8) + (-3)(z-9) = 0$ or $\boxed{3x - 3z = -3}$

Name ______________________

8. *(18 points)* Plane P_1 is given by the equation $3x + y = 6$ and plane P_2 is given by the equation $x + 6z = 3$. Find parametric equations for the line of intersection of the two planes.

We find two points on the line of intersection.

First, put $x=0$ (2). Then $3x+y=6 \Rightarrow y=6$ and $x+6z=3 \Rightarrow z=\frac{1}{2}$. So $(0,6,\frac{1}{2})$ is on the line. (2)

Next, put (2) $y=0$. Then $3x+y=6 \Rightarrow x=2$ and $x+6z=3 \Rightarrow z=\frac{1}{6}$. (2) So $(2,0,\frac{1}{6})$ is on the line.

Therefore $\langle 2-0, 0-6, \frac{1}{6}-\frac{1}{2}\rangle$ (2) $= \langle 2,-6,-\frac{1}{3}\rangle$ (2) is parallel to the line,

and so parametric equations for the line are

$$x = 0 + 2t$$
$$y = 6 + (-6)t \quad (6)$$
$$z = \tfrac{1}{2} + (-\tfrac{1}{3})t.$$

9. *(12 points)* A surface is given in rectangular coordinates by the equation $z^2 = x^2 + y^2$.

a) Give an equation of the surface in cylindrical coordinates, simplifying your answer as much as possible.

[6] $z^2 = (r\cos\theta)^2 + (r\sin\theta)^2$ (3) $\Rightarrow z^2 = r^2(\cos^2\theta + \sin^2\theta)$ (3) $\Rightarrow z^2 = r^2 \Rightarrow \boxed{z = \pm r}$.

[6] b) Given an equation of the surface in spherical coordinates, simplifying your answer as much as possible.

$(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ (3)

$\rho^2\sin^2\phi = \rho^2\cos^2\phi(\cos^2\theta + \sin^2\theta)$

$\cancel{\rho^2}\sin^2\phi = \cancel{\rho^2}\cos^2\phi$ (2)

$\sin^2\phi = \cos^2\phi \Rightarrow \tan^2\phi = 1 \Rightarrow \tan\phi = 1 \Rightarrow \boxed{\phi = \frac{\pi}{4}}$ (1).

10. *(18 points)* Use the integral test to determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is convergent.

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx \ (2) = \lim_{b\to\infty}\int_2^b \frac{1}{x\sqrt{\ln x}}\,dx \ (2) = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{1}{\sqrt{u}}\,du = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} u^{-\frac{1}{2}}\,du \ (2)$$

$$\boxed{\begin{array}{l} u = \ln x \ (2) \\ du = \frac{1}{x}\,dx \ (2) \end{array}}$$

$$= \lim_{b\to\infty}\left[2u^{1/2}\right]_{\ln 2}^{\ln b} \ (2) = \lim_{b\to\infty}\left[2\sqrt{\ln b} - 2\sqrt{\ln 2}\right] \ (2) = \infty \ (2),$$

so the series diverges (2) by the Integral Test.

 Name ______________________

11. *(24 points)* Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{(-3)^n(5n+1)}.$$

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(-3)^{n+1}(5(n+1)+1)}}{\frac{x^n}{(-3)^n(5n+1)}}\right| \textcircled{2} = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\cdot\frac{3^n}{3^{n+1}} \textcircled{2} \cdot\frac{5n+1}{5n+6} = \lim_{n\to\infty}\frac{|x|}{3}\left(\frac{5n+1}{5n+6}\right) \textcircled{2} =$$

$$= \frac{|x|}{3} \textcircled{2} \lim_{n\to\infty}\frac{5n+1}{5n+6} = \frac{|x|}{3}\cdot\lim_{n\to\infty}\frac{5+\frac{1}{n}}{5+\frac{6}{n}} = \frac{|x|}{3}\cdot\left(\frac{5+0}{5+0}\right) = \frac{|x|}{3} \textcircled{2}.$$

So by the Ratio Test, the series converges when $\frac{|x|}{3}<1$, or when $|x|<3$, and diverges when $|x|>3$. ②

When $x=3$ the series is $\sum_{n=1}^{\infty}\frac{3^n}{(-3)^n(5n+1)}$ ② $=\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$ ②, which converges (by Alt. Series Test) ②

When $x=-3$ the series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{(-3)^n(5n+1)}$ ② $=\sum_{n=1}^{\infty}\frac{1}{5n+1}$. Using the limit comparison test, comparing to $\sum\frac{1}{n}$, we get $\lim_{n\to\infty}\frac{\left(\frac{1}{5n+1}\right)}{\frac{1}{n}}$ ② $=\lim_{n\to\infty}\frac{n}{5n+1}=\lim_{n\to\infty}\frac{1}{5+\frac{1}{n}}=\frac{1}{5}\neq 0$

So since $\sum\frac{1}{n}$ diverges, then $\sum\frac{1}{5n+1}$ also diverges. ② The interval of convergence is thus $(-3, 3]$.

12. *(12 points)* Suppose $f(x) = \dfrac{\sin x}{x}$.

[6] a) Find a power series representation for $f(x)$. You do not have to give a formula for the terms in the series, but you should give at least the first four terms of the series, not including zero terms. (Note: for this function, using the formula for the Maclaurin series would be too complicated; use another method instead.)

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ ③, then

③ $$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots.$$

[6] b) Use your answer to part a) to find a series whose sum is $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$. (Give at least the first four terms of the series.)

$$\int_0^1\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots\right)dx \textcircled{2} = \left[x - \frac{x^3}{3}\cdot\frac{1}{3!} + \frac{x^5}{5}\cdot\frac{1}{5!} - \frac{x^7}{7}\cdot\frac{1}{7!} + \cdots\right]_0^1 \textcircled{2}$$

$$= 1 - \frac{1}{3}\cdot\frac{1}{3!} + \frac{1}{5}\cdot\frac{1}{5!} - \frac{1}{7}\cdot\frac{1}{7!} + \cdots \textcircled{2}$$