Quiz 4

Name: Answer key

1. Determine (with explanation) whether the series is absolutely convergent, conditionally convergent, or divergent.

[6] a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+4}$. Since $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+4} = \lim_{n\to\infty} \frac{n/n^2}{1+\frac{4}{n^2}} = \frac{0}{1+0} = 0$ (1)

and $\frac{n}{n^2+4}$ is decreasing for $n \geq 2$ (you can check this by verifying $\frac{d}{dx}\left(\frac{x}{x^2+4}\right) < 0$ for $x > 2$)

then the series converges by the Alt. Series Test. (2)

For $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4}$, we compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. (1) Since $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{n^2+4}}{\left(\frac{1}{n}\right)}$ (1) $\displaystyle= \lim_{n\to\infty} \frac{n^2}{n^2+4}$

$\displaystyle= \lim_{n\to\infty} \frac{1}{1+4/n^2} = \frac{1}{1+0} = 1$, (1) and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges by the Limit Comparison Test. So $\sum (-1)^n \frac{n}{n^2+4}$ converges conditionally.

[5] b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{2^n}$

Put $a_n = (-1)^n \frac{n}{2^n}$.

Then $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (1) $\displaystyle= \lim_{n\to\infty} \left| \frac{\left(\frac{n+1}{2^{(n+1)}}\right)}{\left(\frac{n}{2^n}\right)} \right|$ (1) $\displaystyle= \lim_{n\to\infty} \left(\frac{2^n}{2^{(n+1)}}\right)\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}$ (1) $\displaystyle\frac{1}{2}\cdot\left(\frac{1+\frac{1}{n}}{1}\right) =$

$\displaystyle= \frac{1}{2}\cdot\left(\frac{1+0}{1}\right) = \frac{1}{2}$. (1) Since $\frac{1}{2} < 1$, the series converges absolutely by the Ratio Test. (1)

[9] **2.** Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$.

Let $a_n = \frac{n^2 x^n}{2^n}$. Then $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (1) $\displaystyle= \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 x^n} \right|$ (1) (1)

$\displaystyle= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{|x|^{n+1}}{|x|^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} \cdot |x|$

$\displaystyle= \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right)^2 \cdot \frac{|x|}{2} = \left(\frac{1+0}{1}\right)^2 \cdot \frac{|x|}{2} = \frac{|x|}{2}$. (1)

So by the Ratio Test the series converges if $\frac{|x|}{2} < 1$, or (1) if $|x| < 2$, and diverges if $\frac{|x|}{2} > 1$, or if $|x| < 2$.

The radius of convergence is therefore 2. (1)

When $x = 2$, then the series is $\displaystyle\sum \frac{n^2 \cdot 2^n}{2^n} = \sum n^2$, (1) which diverges by the Divergence Test.

When $x = -2$, then the series is $\displaystyle\sum \frac{n^2(-2)^n}{2^n} = \sum (-1)^n n^2$, (1) which diverges by the Div. Test. So the interval of convergence is (1) $(-2, 2)$.