**EXAM 1**
**Math 2443**
**9-23-11**

Name Key

**Instructions** *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(8 points)* If $f(x,y) = (\cos(3x - y^2))^2$, find

[4] a. $\frac{\partial f}{\partial x}$.

$$= 2\left(\cos(3x-y^2)\right)^1 \cdot \frac{\partial}{\partial x}\left(\cos(3x-y^2)\right) \quad 1$$
$$= 2\cos(3x-y^2)\cdot\left(-\sin(3x-y^2)\right)\cdot\frac{\partial}{\partial x}(3x-y^2) \quad 1$$
$$= \boxed{2\cos(3x-y^2)\left(-\sin(3x-y^2)\right)\cdot 3} \quad 2$$

[4] b. $\frac{\partial f}{\partial y}$.

$$= 2\left(\cos(3x-y^2)\right)^1 \cdot \frac{\partial}{\partial y}\left(\cos(3x-y^2)\right) \quad 1$$
$$= 2\cos(3x-y^2)\cdot\left(-\sin(3x-y^2)\right)\frac{\partial}{\partial y}(3x-y^2) \quad 1$$
$$= \boxed{2\cos(3-y^2)\cdot\left(-\sin(3x-y^2)\right)\cdot(-2y)} \quad 2$$

2. *(16 points)* Suppose $z$ is given implicitly as a function of $x$ and $y$ by the equation

$$yz - \ln z - x - x^2y = 0.$$

[8] a. Find $\frac{\partial z}{\partial x}$. One way is to use the formula $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$ 2, where $F(x,y,z) = yz - \ln z - x - x^2y$ 2. This gives

$$\frac{\partial z}{\partial x} = \frac{-(-1-2xy)}{y - 1/z} = \boxed{\frac{1+2xy}{y-1/z}} \quad 4$$

Another way is to start from $\frac{\partial}{\partial x}(yz - \ln z - x - x^2y) = 0 \Rightarrow y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} - 1 - 2xy = 0$, and then solve for $\frac{\partial z}{\partial x}$.

[8] b. Find $\frac{\partial z}{\partial y}$. As above, we can either write

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \boxed{\frac{-(z - x^2)}{y - 1/z}} \quad 4$$

4, or write

$$\frac{\partial}{\partial y}(yz - \ln z - x - x^2y) = 0 \Rightarrow y\frac{\partial z}{\partial y} + 1\cdot z - \frac{1}{z}\frac{\partial z}{\partial y} - x^2 = 0,$$

and then solve for $\frac{\partial z}{\partial y}$.

3. *(16 points)* Find the equation of the tangent plane to the surface $e^{xz} + yz = 3$ at the point $(0, 1, 2)$.

The surface has equation $F(x,y,z) = 3$ where $F(x,y,z) = e^{xz} + yz$. It is a level surface for $F$, so its tangent plane at $P(0,1,2)$ has equation (6) $(F_x|_P)(x-0) + (F_y|_P)(y-1) + (F_z|_P)(z-2) = 0$.

Since $F_x = e^{xz} \cdot \frac{\partial}{\partial x}(xz) = e^{xz} \cdot z$ (2), $F_y = z$ (2), and $F_z = e^{xz} \cdot x + y$ (2), the equation is (4)

$$e^0 \cdot 2\,(x-0) + 2(y-1) + (e^0 \cdot 0 + 1)(z-2) = 0, \quad \text{or} \quad \boxed{2x + 2(y-1) + (z-2) = 0}$$

4. *(16 points)* Find the directional derivative of $f(x, y, z) = x^3 - xy^2 - z$ at the point $P(1, 1, 0)$ in the direction of the vector $\mathbf{v} = \langle 2, -3, 6 \rangle$.

A unit vector $\vec{u}$ in the direction of $\vec{v}$ is (2)

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \text{ (2)} = \frac{\langle 2,-3,6 \rangle}{\sqrt{4+9+36}} = \left\langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \right\rangle. \text{ (4)}$$

(2)

The gradient of $f$ at $P$ is $\vec{\nabla} f|_P = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\Big|_P = \langle 3x^2 - y^2, -2xy, -1 \rangle|_P$

$= \langle 3\cdot1^2 - 1^2, -2\cdot1\cdot1, -1 \rangle = \langle 2, -2, -1 \rangle$. (2) The directional deriv is

$$\vec{\nabla} f|_P \cdot \vec{u} = \langle 2,-2,-1 \rangle \cdot \left\langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \right\rangle = 2\cdot\frac{2}{7} + (-2)\left(\frac{-3}{7}\right) + (-1)\cdot\frac{6}{7}$$

(2)

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \boxed{4/7}$$

(2)

5. *(12 points)* Suppose $f(x, y)$ has derivatives $f_x = 3$ and $f_y = 5$ at the point $P(1, 2)$. If $f(1, 2) = 7$, estimate the value of $f(1.01, 1.99)$. Explain your answer.

~~Δf at f(1.01,1.99)~~

Take $x = 1$, $y = 2$ (2), $\Delta x = +0.01$, $\Delta y = -0.01$. (2)

Then $f(1.01, 1.99) = f(x+\Delta x, y+\Delta y)$
$f(1,2) = f(x,y)$

$$\Delta z = \text{(2)}\ f(x+\Delta x, y+\Delta y) - f(x,y) \cong f_x \cdot \Delta x + f_y \cdot \Delta y \text{ (2)}$$

$$f(1.01, 1.99) - f(1,2) \cong 3\cdot(0.01) + 5\cdot(-0.01) \text{ (2)}$$

$$f(1.01, 1.99) - 7 \cong 0.03 - 0.05$$

$$f(1.01, 1.99) \cong 7 + 0.03 - 0.05$$

$$f(1.01, 1.99) \approx \boxed{6.98} \text{ (2)}$$

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6. *(16 points)*

[8] **a.** Find the critical point of the function $f(x, y) = 2\ln x + \ln y - 4x - y$.

At the critical point, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, so

$2 \cdot \frac{1}{x} - 4 = 0$ (2) and $\frac{1}{y} - 1 = 0$. (2) So $\boxed{y = 1 \text{ and } x = \frac{1}{2}}$ (4)

The critical point is $(\frac{1}{2}, 1)$.

[8] **b.** Use the second partial derivatives of $f$ to determine whether the critical point is a local maximum, local minimum, or saddle point. Show all work!

(Note: Find $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical point. The point is a maximum if $D > 0$ and $f_{xx} < 0$, a minimum if $D > 0$ and $f_{yy} > 0$, and a saddle point if $D < 0$.)

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{2}{x} - 4\right) = \frac{-2}{x^2} = -2/(1/2)^2 = -8$ (2)

$f_{yy} = \frac{\partial}{\partial y}\left(\frac{1}{y} - 1\right) = -1/y^2 = -1/1^2 = -1$ (2)

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{2}{x} - 4\right) = 0.$ (2)

So $D = (-8)(-1) - 0 = 8$. Since $D = 8 > 0$ and $f_{xx} = -8 < 0$,

The critical point is a maximum. (2)

7. *(16 points)* Use the method of Lagrange multipliers to find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to the constraint $4x + 2y - 5 = 0$.

Let $g(x, y) = 4x + 2y - 5$. (2)

The Lagrange multiplier equations are

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \end{cases} \Rightarrow \begin{cases} 2x = \lambda \cdot 4 & (4) \\ 2y = \lambda \cdot 2 & (4) \end{cases}$$

So $x = 2\lambda$ and $y = \lambda$. Then $4x + 2y - 5 = 0$

gives $4 \cdot 2\lambda + 2 \cdot \lambda - 5 = 0$, or $10\lambda - 5 = 0$, (2)

so $\lambda = \frac{1}{2}$ and $\boxed{x = 2\frac{1}{2} = 1, \; y = \frac{1}{2}}$. (4) The minimum value

is $f(1, \frac{1}{2}) = 1^2 + \left(\frac{1}{2}\right)^2 = \boxed{\frac{5}{4}}$.