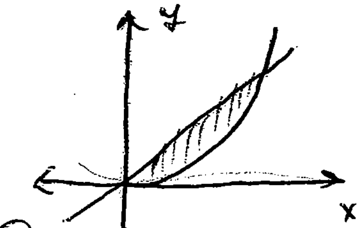
EXAM 2
Math 2443
10-28-11

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. (14 *points*) The region D lies between the line $y = x$ and the curve $y = x^2$ (see diagram). Find the volume which lies above D and below the plane $z = x + y$.

$x = x^2$
$1 = x, \ 0 = x$



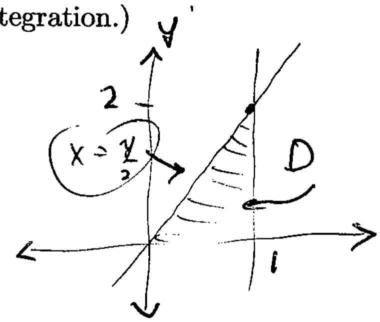
$$V = \int_0^1 \int_{x^2}^{x} x + y \, dy \, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=x^2}^{y=x} dx = \int_0^1 \left[\left(x^2 + \frac{x^2}{2}\right) - \left(x^3 + \frac{x^4}{2}\right) \right] dx =$$

$$= \int_0^1 \left[\frac{3x^2}{2} - x^3 - \frac{x^4}{2} \right] dx = \left[\frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10} \right]_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{1}{4} - \frac{1}{10}$$

$$\boxed{= \frac{3}{20}.}$$

2. (14 *points*) Evaluate $\int_0^2 \int_{y/2}^1 e^{(x^2)} \, dx \, dy$. (Hint: first change the order of integration.)



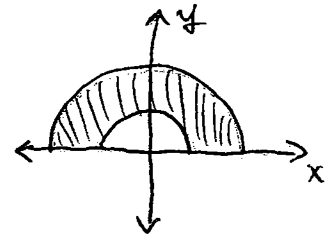
$x = y/2 \Rightarrow y = 2x$

$$= \int_0^1 \int_0^{2x} e^{(x^2)} \, dy \, dx$$

$$= \int_0^1 \left[y e^{x^2} \right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 2x e^{x^2} \, dx = \left[e^{x^2} \right]_0^1 = e^1 - e^0 = \boxed{e - 1.}$$

3. (18 *points*) The region D lies between the circles $x^2+y^2=1$ and $x^2+y^2=4$ and above the x-axis (see diagram.) Evaluate $\int\int_D y\,dA$ by changing to polar coordinates.



$$\int_0^{\pi}\int_1^2 y\ r\,dr\,d\theta$$

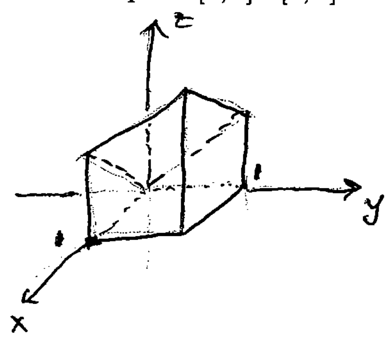
$$= \int_0^{\pi}\int_1^2 r\sin\theta\ r\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\left(\int_1^2 r^2\,dr\right)d\theta = \int_0^{\pi}\sin\theta\left[\frac{r^3}{3}\right]_1^2 d\theta$$

$$= \frac{7}{3}\int_0^{\pi}\sin\theta\,d\theta = \frac{7}{3}\left[-\cos\theta\right]_0^{\pi} = \frac{7}{3}\left[-\cos\pi+\cos 0\right]$$

$$= \frac{7}{3}\left[+1+1\right] = \boxed{\frac{14}{3}.}$$

4. (18 *points*) The solid E has unit density and lies under the plane $z=x+y$ and above the square $[0,1]\times[0,1]$ (see diagram). Find the moment $M_{yz}=\int\int\int_E x\,dV$.



$$M_{yz} = \int_0^1\int_0^1\int_0^{x+y} x\,dz\,dx\,dy$$

$$= \int_0^1\int_0^1 \left[xz\right]_{z=0}^{z=x+y} dx\,dy$$

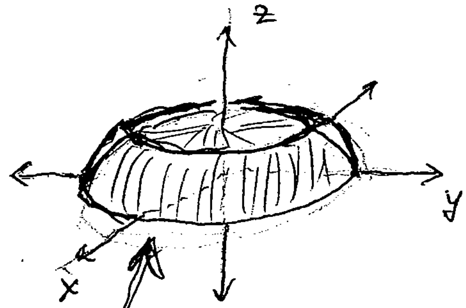
$$= \int_0^1\int_0^1 x(x+y)\,dx\,dy = \int_0^1\int_0^1 (x^2+xy)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3}+\frac{x^2y}{2}\right]_0^1 dy = \int_0^1\left[\frac{1}{3}+\frac{y}{2}\right]dy$$

$$= \left[\frac{y}{3}+\frac{y^2}{4}\right]_0^1 = \frac{1}{3}+\frac{1}{4} = \boxed{\frac{7}{12}.}$$

5. (~~??~~ 18 *points)* Evaluate $\iiint_E z^2\, dV$, where E is the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$, and below the plane $z = 1$. (See diagram.) Use cylindrical coordinates.

$$(1)\int_0^{2\pi} (2)\int_0^1 z^2 \int_0^{\sqrt{4-z^2}} (2)\; r\,dr\,dz\,d\theta \quad (2)$$



(2) $\boxed{r^2+z^2=4 \quad r=\sqrt{4-z^2}}$

$$= \int_0^{2\pi}\int_0^1 z^2 \left[\frac{r^2}{2}\right]_0^{\sqrt{4-z^2}} dz\,d\theta \quad (1)$$

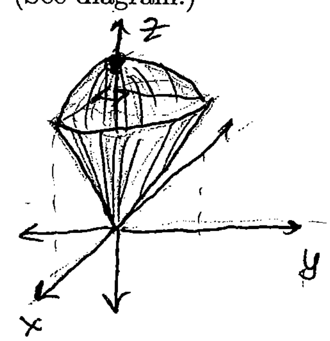
$$= \int_0^{2\pi}\int_0^1 z^2\, \frac{(4-z^2)}{2}\, dz\,d\theta \quad (2)$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}(4z^2-z^4)\,dz\,d\theta \ (2) = \int_0^{2\pi} \frac{1}{2}\left[\frac{4z^3}{3}-\frac{z^5}{5}\right]_0^1 d\theta \quad (2)$$

$$= \int_0^{2\pi} \frac{1}{2}\left[\frac{4}{3}-\frac{1}{5}\right] d\theta \ (2) = \int_0^{2\pi} \frac{17}{30}\,d\theta = \frac{17}{30}\cdot 2\pi \ (2) = \frac{34\pi}{30} = \boxed{\frac{17\pi}{15}}$$

6. (~~??~~ 18 *points)* Evaluate $\iiint_E (x^2+y^2+z^2)\, dV$, where E is the solid that lies within the sphere $x^2+y^2+z^2 = 1$, above the cone $\phi = \pi/3$, and in the first octant (where $x > 0$, $y > 0$, and $z > 0$). (See diagram.)

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{3}} (4) \int_0^1 \rho^2 \,(2)\, \left(\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta\right) \ (2) =$$



$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{3}} \sin\varphi\left(\int_0^1 \rho^4\, d\rho\right) d\varphi\, d\theta \quad (2)$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{3}} \sin\varphi \left[\frac{\rho^5}{5}\right]_0^1 d\varphi\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{5}\left(\int_0^{\frac{\pi}{3}} \sin\varphi\, d\varphi\right) d\theta \ (2) =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{5}\left[-\cos\varphi\right]_0^{\frac{\pi}{3}} d\theta = \frac{1}{5}\int_0^{\frac{\pi}{2}}\left[-\cos\frac{\pi}{3}+\cos 0\right] d\theta \quad (2)$$

$$= \frac{1}{5}\int_0^{\frac{\pi}{2}}\left[1-\frac{1}{2}\right] d\theta \ (2) = \frac{1}{5}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{10}\cdot\frac{\pi}{2} \ (2) = \boxed{\frac{\pi}{20}}$$