EXAM 3 Name *key*
Math 2443
11-18-11

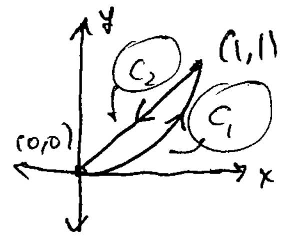
Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(17 points)*

a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = y^2\mathbf{i} - x\mathbf{j}$ and C consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segment $y = \cancel{x}x$ from $(1,1)$ to $(0,0)$.

$\boxed{14}$

Parametrize C_1 as: $\begin{cases} x = t \\ y = t^2 \end{cases}$, $0 \le t \le 1$



and C_2 as: $\begin{cases} x = t \\ y = t \end{cases}$ from $t = 1$ to $t = 0$.

$$\text{Then } \int_C F \cdot dr = \int_{C_1} y^2\,dx - x\,dy + \int_{C_2} y^2\,dx - x\,dy$$

2 2 2 2 2 2

$$= \int_0^1 (t^2)^2\,dt - t(2t\,dt) + \int_1^0 t^2\,dt - t\,dt$$

2

$$= \int_0^1 (t^4 - 2t^2)\,dt - \int_0^1 (t^2 - t)\,dt = \left[\frac{t^5}{5} - \frac{2t^3}{3} - \frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 \quad 1$$

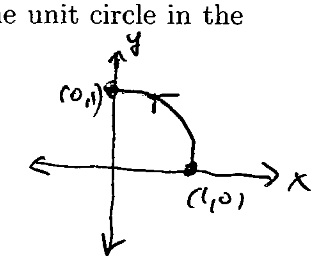
$$= \frac{1}{5} - 1 + \frac{1}{2} = -\frac{4}{5} + \frac{1}{2} = \boxed{-\frac{3}{10}}$$

b) Explain how you can tell from the value of the line integral in a ~~tells you~~ that $\mathbf{F}$ is not conservative.

$\boxed{3}$ If $\vec{F}$ were conservative, then $\int_C \vec{F} \cdot d\vec{r}$ would equal zero, (3) since C is closed. But $\int_C \vec{F} \cdot d\vec{r} \ne 0$, so $\vec{F}$ is not conservative.

2. *(13 points)* Evaluate the line integral $\int_C xy^2\,dx + y^4\,dy$, where C is the portion of the unit circle in the first quadrant, starting at $(1,0)$ and ending at $(0,1)$ (see diagram).

Parametrize C by $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ (2), $0 \le t \le \frac{\pi}{2}$.



1 1 2

$$\text{Then } \int_C xy^2\,dx + y^4\,dy = \int_0^{\frac{\pi}{2}} \cos t \cdot \sin^2 t\,(-\sin t\,dt) + \sin^4 t \cdot \cos t\,dt$$

1 1 2

$$= \int_0^{\frac{\pi}{2}} (-\sin^3 t + \sin^4 t)\cos t\,dt = \int_0^1 (-u^3 + u^4)\,du = \left[\frac{-u^4}{4} + \frac{u^5}{5}\right]_0^1$$

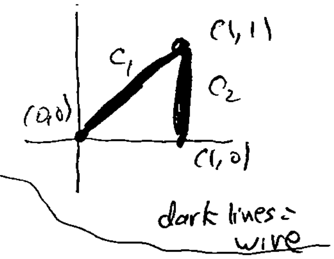
$u = \sin t$ (1)
$du = \cos t\,dt$ (1)

1 1

$$= \frac{-1}{4} + \frac{1}{5} = \boxed{\frac{-1}{20}}.$$

1

3. *(15 points)* A wire of unit density is bent into the shape of the line segment from $(0,0)$ to $(1,1)$ attached to the line segment from $(1,1)$ to $(1,0)$ (see diagram).



a) Find the length L of the wire (you don't need calculus for this).

(3)

$$L = \underbrace{\sqrt{1^2+1^2}}_{\text{segment from }(0,0)\text{ to }(1,1)} + \underbrace{1}_{\text{segment from }(1,1)\text{ to }(1,0)} = \sqrt{2}+1$$

b) Find the y-coordinate of the center of mass of the wire. Use the formula $\bar{y} = \frac{1}{L}\int_C y\, ds$.

[12]

$$\int_C y\, ds = \int_{C_1} y\, ds + \int_{C_2} y\, ds = \int_0^1 t\sqrt{2}\, dt + \int_0^1 t\, dt = \int_0^1 (\sqrt{2}+1)t\, dt$$

(2)

On C_1: $x=t$, $y=t$, $ds = \sqrt{1^2+1^2}\,dt = \sqrt{2}\,dt$, $0 \le t \le 1$ (3)

$$= \left[(\sqrt{2}+1)\frac{t^2}{2}\right]_0^1 \quad (2)$$

On C_2: $x=1$, $y=t$, $ds = \sqrt{0^2+1^2}\,dt = dt$, $0 \le t \le 1$ (3)

$$= (\sqrt{2}+1)\cdot\frac{1}{2} \quad (1)$$

So $\bar{y} = \frac{1}{\sqrt{2}+1}\cdot(\sqrt{2}+1)\cdot\frac{1}{2} = \boxed{\frac{1}{2}}$. (1)

4. *(20 points)* Suppose $\mathbf{F} = (2x+e^y)\mathbf{i} + (xe^y + \ln z)\mathbf{j} + (y/z+3)\mathbf{k}$.

a. Find a function f such that $\mathbf{F} = \nabla f$.

[15]

$\frac{\partial f}{\partial x} = 2x + e^y$ (2) $\Rightarrow f = x^2 + xe^y + C(y,z)$ (2) $\Rightarrow \frac{\partial f}{\partial y} = xe^y + \frac{\partial C}{\partial y}$ (2)

$\frac{\partial f}{\partial y} = xe^y + \ln z \Rightarrow xe^y + \frac{\partial C}{\partial y} = xe^y + \ln z$ (2) $\Rightarrow \frac{\partial C}{\partial y} = \ln z$

$\Rightarrow C = y\ln z + D(z)$ (2)

$\Rightarrow f = x^2 + xe^y + y\ln z + D(z)$.

$\frac{\partial f}{\partial z} = \frac{y}{z} + 3 \Rightarrow y\cdot\frac{1}{z} + \frac{dD}{dz} = \frac{y}{z}+3$ (2) $\Rightarrow \frac{dD}{dz} = 3 \Rightarrow D = 3z + E$ (2)

$\Rightarrow \boxed{f = x^2 + xe^y + y\ln z + 3z + E \text{ (where } E \text{ is any constant)}}$ (1)

[5] **b.** Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is any curve which starts at $(3,1,e)$ and ends at $(e,0,1)$.

By the Fund. Th. of line integrals, $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f \cdot dr =$ (2)

$= f(e,0,1) - f(3,1,e)$ (2) $= [e^2 + ee^0 + e\cdot\ln 1 + 3\cdot 1] - [3^2 + 3e^1 + 1\ln e + 3e]$

$= (e^2 + e + 0 + 3) - [9 + 3e + 1 + 3e]$

$= \boxed{e^2 - 5e - 7}$ (1)

5. *(5 points)* Show that $\mathbf{F} = x^3y^4\mathbf{i} + x^5y^2\mathbf{j}$ is not a gradient vector field.

If $\vec{F}$ were a gradient vector field, $\vec{F} = P\vec{i} + Q\vec{j}$ would have to satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. But here $P = x^3y^4$ and $Q = x^5y^2$, so $\frac{\partial P}{\partial y} = 4x^3y^3 \neq \frac{\partial Q}{\partial x} = 5x^4y^2$. So $\vec{F}$ cannot be a gradient vector field.

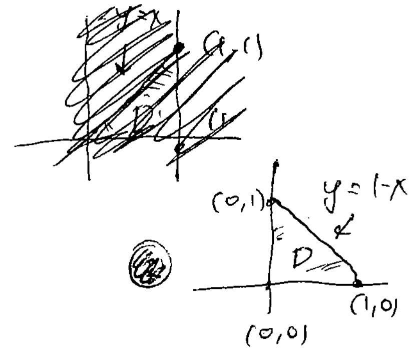
6. *(5 points)* Suppose the region D is the interior of a curve in the plane, f is a function defined on D, and $\nabla f = P\mathbf{i} + Q\mathbf{j}$. What can you say about the value of $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$, and why?

Since $\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$, Then $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0$.

So $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right) dA = \iint_D 0 \, dA = 0$. (There are other ways to answer this question.)

7. *(25 points)* Use Green's Theorem to evaluate the line integral along the given positively oriented curve:

a. $\int_C 3y\,dx + x^3\,dy$, where C is the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$.



[14] Here $P = 3y$, $Q = x^3$, and D is The interior of the triangle, so

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (3x^2 - 3)\,dy\,dx$$

$$= \int_0^1 \left[(3x^2-3)y\right]_0^{(1-x)} dx$$

$$= \int_0^1 (3x^2-3)(1-x)\,dx = \int_0^1 (3x^2 - 3 - 3x^3 + 3x)\,dx$$

$$= \left[x^3 - 3x - \frac{3x^4}{4} + \frac{3x^2}{2}\right]_0^1 = 1 - 3 - \frac{3}{4} + \frac{3}{2} = -2 + \frac{3}{4} = \boxed{-5/4}.$$

b. $\int_C (\sin x + y^3)\,dx + (\cos y - x^3)\,dy$, where C is the circle $x^2 + y^2 = 1$.

[11] Here $P = \sin x + y^3$, $Q = \cos y - x^3$, so $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$

$$= \iint_D (-3x^2 - 3y^2)\,dA = -3\iint_D (x^2+y^2)\,dA,$$ (where D is the interior of the unit circle)

$$= -3\int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta = (-3)\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$$

$$= -3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta = (-3)\int_0^{2\pi}\frac{1}{4}\,d\theta = -\frac{3\cdot 2\pi}{4} = \boxed{\frac{-3\pi}{2}}.$$