FINAL EXAM
Math 2443
12-12-11

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(15 points)* For the function $f(x,y,z) = \ln(2x+3y+6z)$ at the point $P(1,1,0)$:

[10] *a)* In what direction does f have the maximum rate of change?

In the direction of $\vec{\nabla} f \; (2) = \left.\frac{\partial f}{\partial x}\right|_P \vec{i} + \left.\frac{\partial f}{\partial y}\right|_P \vec{j} + \left.\frac{\partial f}{\partial z}\right|_P \vec{k} \; (2) =$

$$= \left[\frac{2}{2x+3y+6z}\vec{i} + \frac{3}{2x+3y+6z}\vec{j} + \frac{6}{2x+3y+6z}\vec{k}\right]\Bigg|_{\substack{x=1\\y=1\\z=0}} \; (4) = \frac{2}{5}\vec{i} + \frac{3}{5}\vec{j} + \frac{6}{5}\vec{k}. \; (2)$$

[5] *b)* What is the maximum rate of change?

It is $|\vec{\nabla} f| \; (2) = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{6}{5}\right)^2} \; (3) = \frac{1}{5}\sqrt{4+9+36} = \boxed{\frac{7}{5}}.$

2. *(15 points)* Find an equation for the tangent plane to the surface $x^2z + y^2z = 1$ at the point $P(1,1,1/2)$.

The surface is a level ~~curve~~ surface for $F(x,y,z) = x^2z + y^2z$, so its tangent plane at P is normal to $\vec{\nabla} F|_P = \left.\frac{\partial F}{\partial x}\right|_P \vec{i} + \left.\frac{\partial F}{\partial y}\right|_P \vec{j} + \left.\frac{\partial F}{\partial z}\right|_P \vec{k}$ (5)

We have $\vec{\nabla} F|_P = \left[2xz\,\vec{i} + 2yz\,\vec{j} + (x^2+y^2)\vec{k}\right]_{\substack{x=1\\y=1\\z=1/2}} \; (5) = 1\vec{i} + 1\vec{j} + 2\vec{k}$,

So an equation for the tangent plane is $\boxed{1(x-1) + 1(y-1) + 2(z-\tfrac{1}{2}) = 0.}$ (5)

3. *(20 points)* Find the minimum value of $f(x,y) = (x-4)^2 + y^2$ subject to the constraint $2x + y = 3$.

At the point $P(x,y)$ where a minimum occurs, we have, for some real number λ,

(3) $\vec{\nabla} f|_P = \lambda \vec{\nabla} g|_P$ where $g(x,y) = 2x+y$.

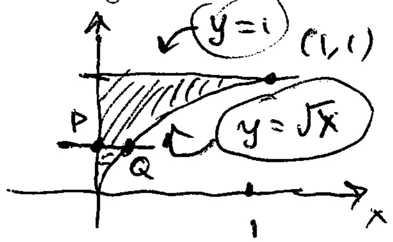
Therefore $2(x-4) = \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} = \lambda \cdot 2$ (4) at P, and also
and $2y = \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} = \lambda \cdot 1$ (4) $\quad 2x+y=3$.

Solving for x and y in terms of λ gives $y = \frac{\lambda}{2}$ and $x = \lambda + 4$, (3)
and substituting into $2x+y=3$ gives $2(\lambda+4) + \frac{\lambda}{2} = 3$, or
$\frac{5\lambda}{2} + 8 = 3$, so $\lambda = \frac{2}{5}(3-8) = -2$. (3) Therefore $y = \frac{-2}{2} = -1$
and $x = -2+4 = 2$. $\boxed{\text{The minimum occurs at } P(2,-1).}$ (3)
and the minimum value is $(2-4)^2 + (-1)^2 = \boxed{5}$.

4. *(20 points)* Evaluate the integral $\int_0^1 \int_{\sqrt{x}}^1 \frac{3}{4+y^3}\, dy\, dx$ by first changing the order of integration.

(5)

The region of integration is shown in the diagram:

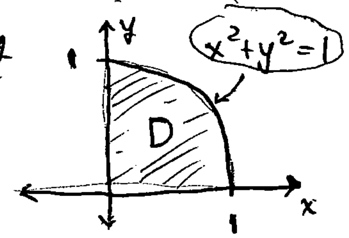


Using horizontal lines like PQ in the diagram we see that the integral equals (5) $\int_0^1 \int_0^{y^2} \frac{3}{4+y^3}\, dx\, dy$, or

$$\int_0^1 \left[\frac{3x}{4+y^3}\right]_{x=0}^{x=y^2} dy \;(2) = \int_0^1 \frac{3y^2}{4+y^3}\, dy \;(2) = \int_4^5 \frac{du}{u} \;(2) = \left[\ln u\right]_4^5 \;(2) = \boxed{\ln 5 - \ln 4} \;(2)$$

$$\left(\begin{array}{l} u = 4+y^3 \\ du = 3y^2\,dy \end{array}\right)$$

5. *(15 points)* Evaluate $\iint_D xy\, dA$, where D is the part of the unit disc which lies in the first quadrant (see diagram).



Using polar coordinates, (2) with $x = r\cos\theta$, $y = r\sin\theta$ (2)

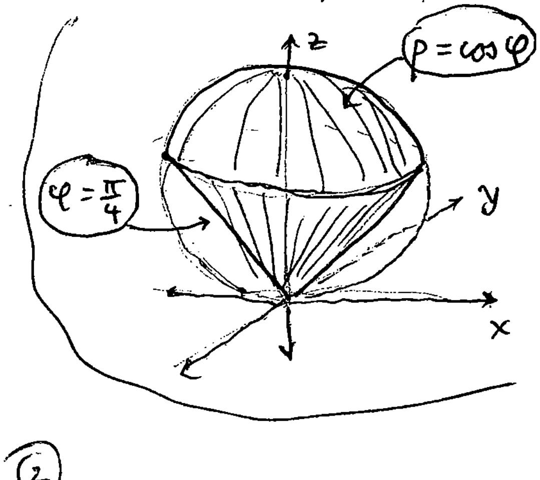
and $dA = r\,dr\,d\theta$, we get

$$(4)\; \int_0^{\frac{\pi}{2}} \int_0^1 (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\int_0^1 r^3\,dr\right)\sin\theta\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^1 \sin\theta\cos\theta\,d\theta \;(2) =$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta \xrightarrow[\substack{u=\sin\theta \\ du=\cos\theta\,d\theta}]{(2)} = \frac{1}{4}\int_0^1 u\,du = \left[\frac{u^2}{8}\right]_0^1 \;(2) = \boxed{\frac{1}{8}.} \;(1)$$

6. *(20 points)* Evaluate $\iiint_E (x^2+y^2+z^2)\, dV$, where E is the solid that lies below the surface $\rho = \cos\phi$ and above the cone $\phi = \pi/4$ (see diagram).



(2) Using spherical coordinates, with (2) $\rho^2 = x^2+y^2+z^2$ and $dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$, we get

$$\int_0^{2\pi}(2) \int_0^{\frac{\pi}{4}}(2) \int_0^{\cos\varphi}(2)\; \rho^2\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\varphi \left[\frac{\rho^5}{5}\right]_{\rho=0}^{\rho=\cos\varphi} d\varphi\,d\theta = (2)$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\varphi\,\frac{\cos^5\varphi}{5}\,d\varphi\,d\theta \;(2) = \int_0^{2\pi}\int_1^{\frac{1}{\sqrt{2}}} \left(-\frac{u^5}{5}\right) du\,d\theta \;(2)$$

$$\left(\begin{array}{l} u = \cos\varphi \\ du = -\sin\varphi\,d\varphi \end{array}\right)$$

$$= \frac{1}{5}\cdot 2\pi \int_{\frac{1}{\sqrt{2}}}^1 u^5\,du \;(2) = \frac{2\pi}{5}\left[\frac{u^6}{6}\right]_{\frac{1}{\sqrt{2}}}^1 \longrightarrow (2) = \frac{\pi}{15}\left[1-\frac{1}{8}\right] = \boxed{7\pi/120}$$

7. *(25 points)* Suppose $\mathbf{F} = (z^2 + 2xy)\mathbf{i} + (x^2 + 1)\mathbf{j} + (2xz - 3)\mathbf{k}$.

[20] *a)* Find a function f such that $\mathbf{F} = \nabla f$.

$\frac{\partial f}{\partial x} = z^2 + 2xy$ (2) $\Rightarrow$ $f = z^2x + x^2y + g(y,z)$ (4)

$\Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y}$ and $\frac{\partial f}{\partial y} = x^2 + 1 \Rightarrow \frac{\partial g}{\partial y} = 1$ (2), so $g = y + h(z)$ (2), and

(2) $f = z^2x + x^2y + y + h(z)$ (2). Therefore $\frac{\partial f}{\partial z} = 2zx + h'(z)$ (2);

and $\frac{\partial f}{\partial z} = 2xz - 3 \Rightarrow h'(z) = -3 \Rightarrow h(z) = -3z + C$ (2), so

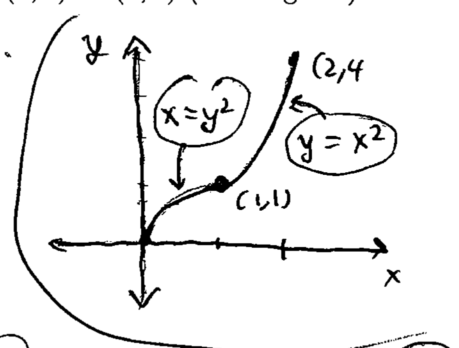
$$\boxed{f = z^2x + x^2y + y - 3z + C}$$ (2) (C can be any constant.)

[5] *b)* Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve which starts at $(1, 2, 3)$ and ends at $(0, 1, 1)$.

By the Fund. Theorem of line integrals, $\int_C \vec{\nabla} f \cdot d\mathbf{r} = f(0,1,1) - f(1,2,3)$
$= (1-3) - (9+2+2-9) = \boxed{-6}$

8. *(25 points)* Evaluate the line integral $\int_C \mathbf{F} \cdot\ d\mathbf{r}$, where $\mathbf{F} = 2x\mathbf{i} + x\mathbf{j}$ and C consists of the arc of the parabola $x = y^2$ from $(0, 0)$ to $(1, 1)$ and the arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ (see diagram).

Let C_1 be the first ~~arc~~ and C_2 the second arc.



We can parameterize C_1 as: (2) $\begin{cases} y = t \\ x = t^2 \end{cases}$, $0 \le t \le 1$,

with (2) $\begin{cases} \frac{dx}{dt} = 2t\,dt \\ \frac{dy}{dt} = 1\,dt \end{cases}$. So $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 2x\,dx + x\,dy =$

$= \int_0^1 2t^2(2t\,dt) + t^2(1\,dt)$ (2) $= \int_0^1 [4t^3 + t^2]dt$ (1) $= \left[t^4 + \frac{t^3}{3}\right]_0^1$ (2) $= \frac{4}{3}$. (2)

We can parameterize C_2 as: $\begin{cases} x = t \\ y = t^2 \end{cases}$ (2) for $1 \le t \le 2$, with

(2) $\begin{cases} \frac{dx}{dt} = 1\,dt \\ \frac{dy}{dt} = 2t\,dt \end{cases}$. So $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_1^2 2t(1\,dt) + t(2t\,dt)$ (2)

$= \int_1^2 [2t + 2t^2]dt$ (1) $= \left[t^2 + \frac{2t^3}{3}\right]_1^2$ (2) $= \left(4 + \frac{16}{3}\right) - \left(1 + \frac{2}{3}\right)$ (2) $= 3 + \frac{14}{3} = \frac{23}{3}$.

So $\int_C \vec{F} \cdot d\vec{r} = \frac{4}{3} + \frac{23}{3} = \frac{27}{3} = \boxed{9.}$ (3)

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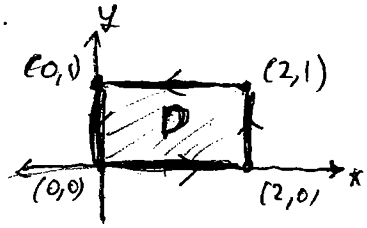
9. *(20 points)* Use Green's theorem to evaluate the line integral $\int_C x\,dx + x^3y\,dy$, where C is the rectangle with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, oriented counterclockwise (see diagram).



Take ~~here~~ $\vec{F} = P\vec{i} + Q\vec{j}$ where $P = x$ and $Q = x^3y$. 2

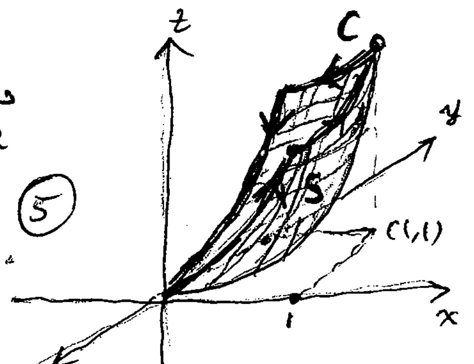
Then $\int_C x\,dx + x^3y\,dy = \int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ 5

(where D is the interior of the rectangle)

$$= \int_0^2 \int_0^1 (3x^2y - 0)\,dy\,dx = \int_0^2 \left[\frac{3x^2y^2}{2}\right]_0^1 dx = \int_0^2 \frac{3x^2}{2}\,dx = \left[\frac{x^3}{2}\right]_0^2 = \boxed{4}$$

5 2 2 2 2

10. *(25 points)* Suppose $\mathbf{F} = 2\mathbf{i} + 3y\mathbf{j} - 3z\mathbf{k}$ and S is the portion of the surface $z = x^2 + y^2$ which lies above the rectangle $\{0 \le x \le 1, 0 \le y \le 1\}$ in the xy-plane, oriented so $\mathbf{n}$ points upwards. Suppose the closed curve C is the boundary of S, oriented as shown in the diagram.



[15] *a)* Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$. We can parameterize S as

$$\begin{cases} x = u \\ y = v \\ z = u^2 + v^2 \end{cases}, \quad \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1 \end{matrix}, \quad \text{so} \quad \vec{r} = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k}$$

and $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$. 5

Then $\iint_S \vec{F}\cdot d\vec{S} = \int_0^1 \int_0^1 (2\vec{i} + 3v\vec{j} - 3(u^2+v^2)\vec{k})\cdot(-2u\vec{i} - 2v\vec{j} + \vec{k})\,du\,dv$ 2

$$= \int_0^1 \int_0^1 (-4u - 6v^2 - 3u^2 - 3v^2)\,du\,dv = \int_0^1\int_0^1 [-4u - 3u^2 - 9v^2]\,du\,dv$$ 2 2

$$= \int_0^1 \left[-2u^2 - u^3 - 9uv^2\right]_0^1 dv = \int_0^1 [-2 - 1 - 9v^2]\,dv = \left[-3v - 3v^3\right]_0^1 = \boxed{-6.}$$ 2

[5] *b)* If $\mathbf{G} = 3yz\mathbf{i} + \mathbf{j} + 2y\mathbf{k}$, find the curl of $\mathbf{G}$.

$$\text{curl}\,\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3yz & 1 & 2y \end{vmatrix} = (2-0)\vec{i} - (0-3y)\vec{j} + (0-3z)\vec{k}$$

$$= \boxed{2\vec{i} + 3y\vec{j} - 3z\vec{k}}$$ (which is $\vec{F}$ in a))

[5] *c)* From the results of parts *a)* and *b)*, what can you say about $\int_C \mathbf{G}\cdot d\vec{r}$? Explain your answer.

By Stokes' Theorem, (since the orientations on C and S agree)

$$\int_C \vec{G}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{G}\cdot d\vec{S} \underbrace{=}_{\text{by b)}} \iint_S \vec{F}\cdot d\vec{S} = -6 \quad \text{(by a)).}$$