Quiz 6

Name: key

1. Let $\mathbf{F}(x,y) = (x-y)\mathbf{i} + x\mathbf{j}$, and let C be the circle of radius one centered at the origin (parametrized counterclockwise).

a. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a line integral.

Let $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then $dx = -\sin t \, dt$, $dy = \cos t \, dt$, and

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (x-y)\,dx + x\,dy = \int_0^{2\pi} (\cos t - \sin t)(-\sin t \, dt) + \cos t(\cos t \, dt) =$$

$$= \int_0^{2\pi} [-\cos t \sin t + \sin^2 t + \cos^2 t]\,dt = \int_0^{2\pi} -\cos t \sin t \, dt + \int_0^{2\pi} 1\,dt =$$

$$\underset{\left(\substack{u = \cos t \\ du = -\sin t}\right)}{=} \int_{u=1}^{u=1} u\,du + \int_0^{2\pi} 1\,dt = \left[\frac{u^2}{2}\right]_1^1 + [t]_0^{2\pi} = \left(\frac{1}{2} - \frac{1}{2}\right) + (2\pi - 0) = \boxed{2\pi}$$

b. Use Green's Theorem to express $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a double integral, and evaluate the double integral.

Here $\vec{F} = P\vec{i} + Q\vec{j}$ with $P = x - y$ and $Q = x$,

so Green's Theorem says $\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

(where D is the interior of the unit circle C)

$$= \iint_D (1 - (-1))\,dA = \iint_D 2\,dA =$$

$$= \int_0^{2\pi} \int_0^1 2\,r\,dr\,d\theta = \int_0^{2\pi} [r^2]_0^1\,d\theta = \int_0^{2\pi} 1\,d\theta$$

$$= \boxed{2\pi}.$$

2. Let $\mathbf{F}(x,y) = (x-y)\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$, and evaluate

a. div $\mathbf{F}$ $= \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 1 + 0 + 0 = 1$

b. curl $\mathbf{F}$ $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y) & x & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z} x\right)\vec{i} - \left(\frac{\partial}{\partial x} 0 - \frac{\partial}{\partial z}(x-y)\right)\vec{j}$

$$+ \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x-y)\right)\vec{k}$$

$$= 0\vec{i} - 0\vec{j} + (1-(-1))\vec{k} = 2\vec{k}.$$