Review for Third Exam

The third exam will cover sections 17.1 through 17.4 of the text.

17.1. This section aims to give you an intuitive understanding of what is meant by a "vector field". Although I didn't assign any homework from this section, it would make this class easier for you if you spent a few minutes on some of the first eighteen problems from the end of this section, drawing pictures and thinking about how the vector fields in the problems behave.

This section also contains the definitions of the terms "conservative vector field" and "gradient vector field". Notice that according to the book's definitions of these terms, they mean exactly the same thing: namely, a vector field which is the gradient of some function.

Related to the concept of a conservative or gradient vector field is that of a vector field whose line integrals are "path independent". We say that a vector field $\mathbf{F}$ defined on a region D has the path independence property if the value of the line integral of $\mathbf{F}$ over a curve C is determined solely by the beginning and end points of C, and does not depend on what path C takes in between these points.

It turns out that conservative (or gradient) vector fields and path independent vector fields are the same thing. More precisely, if a vector field is a conservative (or gradient) vector field, then it has the path independence property, and if a vector field has the path independence property, then it is a conservative (or gradient) vector field. This is proved in section 17.3 of the text.

17.2. This section defines several types of "line integrals", which are integrals of functions or vector fields over curves in the plane or in space. Remember that by a "curve" we mean a subset of the plane which can be described by parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$; or a subset of space which can be described by parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$, $a \le t \le b$. (In particular, by this definition a straight line segment is a curve.) You should be familiar with the parametric representations of simple curves; these were described in the text in sections 11.1, 11.2, and 14.1. Next week, after this test is over, we'll be studying the parametric representations of two-dimensional surfaces.

The different kinds of line integral are evaluated by using the parametric equations for the curve C to substitute functions of t for x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ and $\frac{ds}{dt}$. For example, see box 3 on page 1070 or box 7 on page 1073. Also, remember that $\int_C \mathbf{F}\cdot\, d\mathbf{r}$ is defined by

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right]\, dt,$$

where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

You should try to get used to thinking of line integrals as limits of sums, just as the integrals we worked with in Calculus II and Calculus III were limits of sums. For example, $\int_C f\, ds$ should be thought of as the limit of a sum in which each term is the product of the value of f at a point on the curve times the length ds of a small segment of the curve (see page 1070). This makes it easier to understand and remember the various situations in which line integrals are used in physics and geometry.

17.3. This section contains two main ideas: first, the fundamental theorem for line integrals (Theorem 2 on page 1082); and second, the relation between conservative vector fields in the plane and plane vector fields which satisfy the equation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Vector fields satisfying that equation are sometimes called "curl-free" vector fields, for reasons that will be explained when we cover section 17.5.

Curl-free vector fields are almost the same thing as conservative vector fields, but not quite. Every conservative vector field in any region in the plane is curl-free. This follows from the equality of mixed partial derivatives, as explained on page 1085. But there are plane vector fields defined in some regions of the plane which are not conservative. An important example is the vector field

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

As we saw in class, this vector field is curl-free on the region consisting of the plane with the point $(0, 0)$ removed. However, it is not conservative, because it's not path independent.

On the other hand, if a vector field is defined on a **simply connected** set and is curl-free on that set, then the vector field is conservative. For plane vector fields, this fact is stated as Theorem 6 on page 1086; and is proved using Green's Theorem. The proof is worth reviewing — I gave it in class; and you can also find it on page 1096 of the text.

You should memorize the fundamental theorem of line integrals and be able to use it. Part of doing this is finding the function f which a given vector field $\mathbf{F}$ is the gradient of. We did several examples of this in class, on the homework, and on Quiz 5.

17.4. You should memorize Green's theorem (in the red box on page 1091), and be able to do problems like the ones worked out in Examples 1, 2, and 3 in this section, and like the ones assigned on the homework for this section. You do not need to worry about the material in the section which comes between the end of Example 3 and the end of Example 5, as we haven't covered it in class yet and it won't be needed for the exam.