Review for Final Exam

The final exam is comprehensive, but will be weighted somewhat towards questions from the latter half of the course. The topics covered will be the same as those on the first three tests, together with the topics described below from sections 17.5, 17.6, 17.7, and 17.8. I've also included a review of what we covered in Chapter 15. You can use the review sheets for the second and third exams to see what's covered from Chapter 16 and Sections 17.1, 17.2, 17.3 and 17.4.

17.5. We've covered this entire section in class. It introduces the curl (which is an operation you perform on a vector field to get another vector field) and the divergence (which is operation your perform on a vector field to get a scalar function). Remember not to confuse the divergence with the gradient, which is an operation you perform on a scalar function to get a vector field (see section 15.6, page 951 and page 949).

A couple of useful facts mentioned about the curl and the divergence in this section are: the curl of a gradient vector field is always zero, and the divergence of a curl is always zero. Also, if a vector field $\mathbf{F}$ is defined on a simply connected set and curl $\mathbf{F} = \mathbf{0}$ on that set, then $\mathbf{F}$ must be a gradient (conservative) vector field (this can be proved using Stokes' Theorem). On the other hand it's possible for a vector field to have zero curl on a non-simply connected set and yet not be conservative. See the discussion on the review sheet for Exam 3.

17.6. This section explains how to parameterize surfaces: that is, how to describe a two-dimensional surface in three-dimensional space by giving the coordinates x, y, and z of points on the surface as functions of two parameters u and v. In general, this can be a complicated task, as you can see by looking at some of the pictures in exercise 18 on page 1115. However, you won't be required to know or to find any fancy parameterizations. You should, though, be familiar with a few simple parameterizations like the ones in Examples 1, 4, 5, 6, and 7 (notice there are two parameterizations given in Example 7 — the first one would be more useful if you had to consider the portion of the surface above a rectangle in the xy-plane, and the second would be more useful if you had to consider the portion of the surface above a circle in the xy-plane).

This section also introduces the following ideas, with which you should be familiar:

Suppose a surface S is given by the parametric equations $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$. Then the position vector of a point on a surface S is

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

and two vectors tangent to the surface at a given point P are

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

and

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k},$$

where the derivatives are evaluated at P.

The vector $\mathbf{r}_u \times \mathbf{r}_v$ is perpendicular to the surface S at P, and has length $|\mathbf{r}_u \times \mathbf{r}_v|$, which is equal to the area of the parallelogram formed by the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

The area of the surface S is given by the integral

$$\int\int_S 1\ dS = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v|\ du\ dv,$$

where D is the region in the uv-plane which includes all the values of (u, v) used to obtain the points on S. This is illustrated in Examples 10 and 11 of this section.

If you're really pressed for time, you can skip reading Examples 3, 8 and 9 of this section, since I won't ask questions exactly like those examples on the exam. (Reading them would aid your understanding of the material, of course.)

Another useful bit of information in this section is that if a surface is given by the equation $z = f(x, y)$ and you decide to parameterize it by taking $u = x$, $v = y$, and $z = f(u, v)$, then you get the formulas

$$\mathbf{r}_u \times \mathbf{r}_v = -\frac{\partial f}{\partial u}\mathbf{i} - \frac{\partial f}{\partial v}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}.$$

(This is stated on page 1113, and again on page 1119.) Of course similar formulas hold for surfaces given by equations of the form $x = f(y, z)$ or $y = f(x, z)$.

17.7. The topic of this section is the integral of a vector field $\mathbf{F}$ over a surface S , sometimes called the flux integral of $\mathbf{F}$ over S, defined by

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ du\ dv.$$

Here the dot on the right hand side between the vectors $\mathbf{F}$ and $\mathbf{r}_u \times \mathbf{r}_v$ is the dot product of vectors, defined in Calculus III (Section 13.3). In computing this integral, you have to keep in mind that it can change sign depending on how the surface is parameterized: if the surface is parameterized in two different ways, and for one parameterization the vector $\mathbf{r}_u \times \mathbf{r}_v$ always has direction opposite to what it has for the other parameterization, then the value of $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ will differ by a factor of -1 between the two parameterizations. (In particular, if you switch u and v the integral will switch sign.)

You should read carefully all the examples that you find time to read. The more you have time to read, the easier you'll find the homework and test problems.

17.8. You should at least know the statement of Stokes' Theorem (page 1129) and review Examples 1 and 2 in this section. You can skip the remainder of the section if you're short on time.

17.9 The divergence theorem will not be covered on the final exam. However, as I mentioned in class, the divergence theorem is an interesting relation between triple integrals (which you already know about from Chapter 156) and the surface integrals you learned about in Section 17.7, and it can't be anything but helpful to know about it.

Chapter 15: Here's a quick list of topics to review from Chapter 15.

15.1: Graphs of functions of two variables, level curves of functions of two variables, level surfaces of functions of three variables.

15.2: You can skip this section.

15.3: Partial derivatives — how to compute them, and their interpretation as slopes of curves on the graph of a function.

15.4: Equation for the tangent plane to the graph of $z = f(x, y)$ (page 928); the differential (bottom of page 932) and its use in linear approximation (Examples 4 and 5)

15.5: Chain rule, implicit differentiation

15.6: Formula for directional derivative on page 948, interpretation as slope of a curve on the graph of a function (the slope of line T in Figure 3 page 946 is the directional derivative of f in the direction $\mathbf{u}$), the definition of the gradient for functions of two variables or functions of three variables, the theorem at top of page 952 relating gradients to directional derivatives, the equation for the tangent plane to the graph of the surface $F(x, y, z) = \text{constant}$ (bottom of page 953).

15.7: How to find local maxima or minima of functions of two variables. You do not need to memorize the second derivative test. I will also not ask questions about absolute maxima and minima, so there will not be s question like Example 7.

15.8: If I include any problem on the material from this section it will be quite simple, like Examples 2 or 4.