EXAM 1
Math 2513
5-31-13

Name key

1. *(15 points)* Show that the statements $(p \wedge q) \to r$ and $\neg p \vee \neg q \vee r$ are logically equivalent.

Method 1: We show the statements are equivalent by constructing a truth table:

p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$\neg p$	$\neg q$	$\neg p \vee \neg q \vee r$
T	T	T	T	T	F	F	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns for $(p \wedge q) \to r$ and $\neg p \vee \neg q \vee r$ are identical, the two propositions are equivalent.

Method 2: The only way $(p \wedge q) \to r$ can be false is for $p \wedge q$ to be true and r to be false. And this can only happen when p and q are both true and r is false, or when $\neg p$, $\neg q$, and r are all false. But this is also the only way $\neg p \vee \neg q \vee r$ can be false. Since the two propositions are false under exactly the same conditions, they are equivalent.

2. *(15 points)* You come across three people, A, B, C. You know that one of them is a knight, who always makes true statements; another is a knave, who always makes false statements, and the third is normal, meaning he sometimes makes true statements and sometimes makes false statements.

A says: I am normal.
B says: What A said is true.
C says: I am not normal.

Say which of the three is the knight, which is the knave, and which is normal. You should prove that your answer is correct.

Since A says he is normal, he cannot be a knight, because a knight would not lie and say he is normal. Therefore A must be either normal or a knave. 3 2

If A is normal, then B is either a knight or a knave. But since B is telling the truth, he must be a knight. 3 So C has to be the knave. But then C is telling the truth, which contradicts him being a knave. This contradiction shows that A cannot be normal. 2

5 Therefore $\boxed{\text{A must be a knave}}$. So A is lying, and hence B's statement is false. So B cannot be the knight, and $\boxed{\text{B must be normal}}$. Then $\boxed{\text{C has to be the knight}}$ ~~the knave~~ ~~and indeed~~ ~~C is lying when he says he is not~~ (and indeed C is telling the truth when he says he is not normal).

3. *(15 points)* Consider the statement

$$\forall x\ \exists y\ (x = 2y + 5)$$

and assume that the domain of both the variables x and y consists of all real numbers $\mathbb{R}$.

[3] a) Express the statement in English.

For every real number x, there exists a real number y such that $x = 2y + 5$.

[3] b) Is the statement true or false? Justify your answer.

True, because for a given x we can take $y = \frac{x-5}{2}$.

Now for parts c) and d), consider the statement

$$\exists y\ \forall x\ (x = 2y + 5)$$

and assume again that the domain consists of all real numbers.

[3] c) Express the statement in English.

There exists a real number y such that every real number x is equal to $2y+5$.

[3] d) Is the statement true or false? Justify your answer.

False; for every real number y we can always find a real number x which does not equal $2y+5$. For example taking $x = 2y+6$ we have $x \neq 2y+5$, because $6 \neq 5$.

[3] e) Write the negation of the statement.

$\forall y\ \exists x\ (x \neq 2y+5)$

4. *(10 points)*

[5] a) Suppose E and F are sets and $E \subseteq F$. Show that $\overline{F} \subseteq \overline{E}$.

1 Suppose $x \in \overline{F}$. Then $x \notin F$ 1 Since $E \subseteq F$, we know that $x \in E \to x \in F$ is true 1 Therefore the contrapositive, $x \notin F \to x \notin E$ is also true. 1 Since $x \notin F$, it follows (modus ponens) that $x \notin E$. 1 So $x \in \overline{E}$. We have proved $x \in \overline{F} \to x \in \overline{E}$, so $\overline{F} \subseteq \overline{E}$.

[5] b) Suppose A and B are sets and $A \subseteq B \cap C$. Show that $\overline{B} \cup \overline{C} \subseteq \overline{A}$.

We are given that $A \subseteq B \cap C$,

3 so from part a) we know that $\overline{B \cap C} \subseteq \overline{A}$.

By de Morgan's laws, $\overline{B \cap C} = \overline{B} \cup \overline{C}$. So $\overline{B} \cup \overline{C} \subseteq \overline{A}$.

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5. *(10 points)* Give an example of a function $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ that is onto but not one-to-one. (Here $\mathbb{Z}$ denotes the set of integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ and $\mathbb{N} \cup \{0\}$ is the set $\{0, 1, 2, 3 \dots\}$.) Prove that your example is correct.

One example is given by $f(n) = |n|$ (That is, $f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n < 0 \end{cases}$. Since $f(-1) = 1 = f(1)$ and $-1 \neq 1$, Then f is not one-to-one. Also, for every $n \in \{0, 1, 2, 3, \dots\}$ we have $f(n) = n$, so the statement $\forall n \in \mathbb{N} \cup \{0\}\ \exists m \in \mathbb{Z}\ f(m) = n$ is true. So f is onto.

Of course, there are many other possible examples.

(4) for a correct definition, (3) for a proof that the function is onto, (3) for a proof that it is not one-to-one.

6. *(10 points)* Suppose $f : A \to B$ and $g : B \to C$ are functions. Prove that if f is one-to-one and g is one-to-one, then $g \circ f : A \to C$ is one-to-one.

Suppose a_1 and a_2 are elements of A and $g \circ f(a_1) = g \circ f(a_2)$. This means that $g(f(a_1)) = g(f(a_2))$. Since g is one-to-one, it follows from the last equation that $f(a_1) = f(a_2)$. But then since f is one-to-one it follows that $a_1 = a_2$.

We have proved that for all a_1 and a_2 in A, $g \circ f(a_1) = g \circ f(a_2) \to a_1 = a_2$. So $g \circ f$ is one-to-one.

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7. *(15 points)* Suppose a_n is a sequence that satisfies the recurrence relation $a_n = 3a_{n-1}$ for $n = 1, 2, 3, \ldots$, and $a_0 = 5$. Find formulas for

a) a_n

[6] We have $a_0 = 5$, $a_1 = 3 \cdot a_0 = 5 \cdot 3$
$a_2 = 3 \cdot a_1 = 5 \cdot 3 \cdot 3$
$a_3 = 3 \cdot a_2 = 5 \cdot 3 \cdot 3 \cdot 3$, etc.

So for all $n \in \mathbb{N}$, $a_n = 5 \cdot 3^n$.

[9] b) $\sum_{k=0}^{n} a_k$ (your answer should be a formula involving only two terms).

You can either remember the formula for the sum of a geometric series, or re-prove it as follows:

Let $S = \sum_{k=0}^{n} a_k = 5 + 5 \cdot 3 + 5 \cdot 3^2 + \ldots + 5 \cdot 3^{n-1} + 5 \cdot 3^n$.

Then $3 \cdot S = 5 \cdot 3 + 5 \cdot 3^2 + \ldots + 5 \cdot 3^n + 5 \cdot 3^{n+1}$,

So $3S - S = 5 \cdot 3^{n+1} - 5$, so $2S = 5 \cdot 3^{n+1} - 5$, so

$$\boxed{S = \frac{5}{2} \cdot 3^{n+1} - \frac{5}{2}}$$

8. *(10 points)*

[5] a) Find a number a in $\{0, 1, 2, 3, 4, 5, 6\}$ such that $a \cdot 4 \equiv 1 \pmod 7$.

$0 \cdot 4 \equiv 0 \pmod 7$
$1 \cdot 4 \equiv 4 \pmod 7$
$2 \cdot 4 \equiv 8 \equiv 1 \pmod 7$ so we can take $a = 2$
⋮

(this is actually the only solution in $\{0, 1, 2, 3, 4, 5, 6\}$)

[5] b) Find a number b in $\{0, 1, 2, 3, 4, 5, 6\}$ such that $b^2 \equiv -5 \pmod 7$.

$0^2 \equiv 0 \pmod 7$
$1^2 \equiv 1 \pmod 7$
$2^2 \equiv 4 \equiv -3 \pmod 7$
$3^2 \equiv 9 \equiv 2 \equiv -5 \pmod 7$ so we can take $b = 3$.

(actually there is also another solution: $b = 4$, since then $b^2 \equiv 16 \equiv 2 \equiv -5 \pmod 7$)