EXAM 2
Math 2513
6-13-13

Name key

1. *(15 points)* Find the binary expansion of the number 2185.

Dividing by 2, we get

| | (remainder) |
|---|---|
| 2185 | 1 |
| 1092 | 0 |
| 546 | 0 |
| 273 | 1 |
| 136 | 0 |
| 68 | 0 |
| 34 | 0 |
| 17 | 1 |
| 8 | 0 |
| 4 | 0 |
| 2 | 0 |
| 1 | 1 |

5 divide by 2 and remainder process

5 getting divisions and remainders right

5 putting remainders in reverse order

So in base 2, $2185 = (100010001001)_2$

2. *(15 points)* Show that if $n$ is an odd integer, then $n(n+2) \equiv 3 \pmod 4$.

Since $n$ is odd, then there exists an integer $k$ such that

3 $n = 2k+1$. So

$$
\begin{aligned}
n(n+2) &= (2k+1)(2k+1+2) \\
&= (2k+1)(2k+3) \quad 3 \\
&= 4k^2 + 2k + 6k + 3 \\
&= 4k^2 + 8k + 3 \quad 3 \\
&= 4(k^2+2k) + 3
\end{aligned}
$$

Therefore, $n(n+2) - 3 = 4(k^2+2k)$. 3

So $n(n+2)-3$ is divisible by 4.

Hence $n(n+2) \equiv 3 \pmod 4$. 3

3. *(15 points)* Find $3^{1126}$ **mod** 20. Show all work.

$3^1 \equiv 3 \pmod{20}$
$3^2 \equiv 9$
$3^3 \equiv 27 \equiv 7$
$3^4 \equiv 21 \equiv 1$
$3^5 \equiv 3$
$3^6 \equiv 9$
$3^7 \equiv 7$
5 $3^8 \equiv 1$, etc.

So the value of $3^k$ is determined by the value of $k$ mod 4. 5

Dividing 4 into 1126 we find

$1126 = 4 \cdot 281 + 2$,

so $1126 \equiv 2$ mod 4.

So $3^{1126} \equiv 3^2 \equiv 9 \pmod{20}$ 5

$$\begin{array}{r} 281 \\ 4\overline{)1126} \\ 8 \\ \hline 32 \\ 32 \\ \hline 06 \\ 4 \\ \hline 2 \end{array}$$

4. *(10 points)*

a) Use the Euclidean algorithm to find the greatest common divisor of 228 and 81.

[6]

$228 = 2 \cdot 81 + 66$ 1

$81 = 1 \cdot 66 + 15$ 1

$66 = 4 \cdot 15 + 6$ 1

$15 = 2 \cdot 6 + \boxed{3}$ ← So $\gcd(228, 81) = 3$. 3

$6 = 2 \cdot 3 + 0$

b) Find the least common multiple of 228 and 81 (showing your work).

[4]

$\operatorname{lcm}(228, 81) = \dfrac{228 \cdot 81}{\gcd(228, 81)} = \dfrac{228 \cdot 81}{3} = (228)(27)$

$= 6156$

$$\begin{array}{r} 228 \\ \times\ 27 \\ \hline 1596 \\ 456 \\ \hline 6156 \end{array}$$

 Name ______________________

5. *(25 points)* Find (showing your work):

a) Integers $s$ and $t$ such that $s \cdot 165 + t \cdot 196 = 1$.

[18]

$196 = 165 + 31$

$165 = 5 \cdot 31 + 10$

$31 = 3 \cdot 10 + 1$ (6)

so $1 = 31 - 3 \cdot 10$ (2)

$= 31 - 3 \cdot (165 - 5 \cdot 31)$ (2)

$= 16 \cdot 31 - 3 \cdot 165$ (2)

$= -3 \cdot 165 + 16 \cdot 31$

$= -3 \cdot 165 + 16 \cdot (196 - 165)$ (2)

$= -19 \cdot 165 + 16 \cdot 196$ (2)

Take $s = -19$ and $t = 16$ (2)

check:

$310 + 186 = 496$, $165 \cdot 3 = 495$

$165 \cdot 19$: $1485 + 165 = 3135$; $196 \cdot 16$: $1176 + 196 = 3136$ ✓

b) A number $a$ such that $a \cdot 165 \equiv 1 \pmod{196}$.

[2]

Since $-19 \cdot 165 + 16 \cdot 196 = 1$, then $-19 \cdot 165 - 1$ is divisible by 196, so $-19 \cdot 165 \equiv 1 \pmod{196}$

Take $a = -19$.

c) A number $b$ such that $b \in \{0, 1, 2, \ldots, 195\}$ and $b \cdot 165 \equiv 1 \pmod{196}$.

[5]

Since the value of $b \cdot 165$ mod 196 is determined by the value of $b$ mod 196, we need only find a number $b \in \{0, 1, 2, \ldots, 195\}$ which is congruent to $a = -19$.

But $-19 \equiv -19 + 196 \equiv 177 \pmod{196}$,

So we can take $\boxed{b = 177}$.

6. (20 points) Use induction to prove that for all positive integers $n$,

$$1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Let $P(n)$ be this equation. (2)

Then $P(1)$ is the equation

$$1^2 + (2\cdot 1+1)^2 = \frac{(1+1)(2\cdot 1+1)(2\cdot 1+3)}{3},$$ (2)

or $1^2 + 3^2 = \frac{2\cdot 3\cdot 5}{3}$

or $1 + 9 = 10$, which is true. (2)

Now assume $P(k)$ is true, so (2)

$$1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}.$$

Adding $(2k+3)^2$ to both sides gives (3)

$$1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 + (2k+3)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2.$$

$$= (2k+3)\left(\frac{(k+1)(2k+1)}{3} + (2k+3)\right)$$ (3)

$$= (2k+3)\left(\frac{(k+1)(2k+1) + 3\cdot(2k+3)}{3}\right)$$

$$= \frac{(2k+3)(2k^2+3k+1+6k+9)}{3}$$ (3)

$$= \frac{(2k+3)(2k^2+9k+10)}{3}.$$

$$= \frac{(2k+3)(k+2)(2k+5)}{3}$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3}.$$ (2)

Scratch work:

$(k+1+1)(2(k+1)+1)(2(k+1)+3)$

$= (k+2)(2k+3)(2k+5)$

$= (2k+3)(k+2)(2k+5)$

$= (2k+3)(2k^2+9k+10)$

So $1^2 + 3^2 + \ldots + (2(k+1)+1)^2 = \frac{(k+1+1)(2(k+1)+1)(2(k+1)+3)}{3}$ (1)

So $P(k+1)$ is true, and the statement is proved by induction.