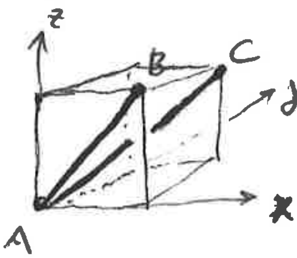
EXAM 1
Math 2934
9-24-15

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* Find the cosine of the angle between the diagonal of a cube and the diagonal of one of its faces (see diagram).



If we take the x, y, and z axes as shown, and assume the cube has side length 1, then A is $(0,0,0)$; B is $(1,0,1)$; and C is $(1,1,1)$. 2

So $\vec{AB} = \langle 1-0, 0-0, 1-0\rangle = \langle 1,0,1\rangle$ and

$\vec{AC} = \langle 1-0, 1-0, 1-0\rangle = \langle 1,1,1\rangle$. 2

The cosine of the angle between them is $\dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|}$ 2 $= \dfrac{1\cdot 1 + 0\cdot 1 + 1\cdot 1}{\sqrt{2}\,\sqrt{3}}$ 2 ,

2

or $\boxed{\dfrac{2}{\sqrt{2}\sqrt{3}}}$ (or $\dfrac{\sqrt{6}}{3}$).

2. *(12 points)* A line passes through the point $(0,2,1)$ and is parallel to the vector $\langle 3,1,-1\rangle$.

a. Write parametric equations for the line.

[6]

$$\begin{cases} x = 0 + 3t \\ y = 2 + 1\cdot t \\ z = 1 + (-1)t \end{cases}, \quad \text{or} \quad \begin{cases} x = 3t \\ y = 2 + t \\ z = 1 - t \end{cases}.$$

b. Decide whether the line lies within the plane $x + 2y + 5z = 9$. Explain the reason for your answer.

[6]

One way is to check whether the equations in part a) imply that $x+2y+5z=9$ for all t. We have, if $\begin{cases} x = 3t \\ y = 2+t \\ z = 1-t \end{cases}$

2

that
$$\begin{aligned} x+2y+5z &= 3t + 2(2+t) + 5(1-t) \\ &= 3t + 4 + 2t + 5 - 5t \quad 2 \\ &= (3+2-5)t + 9 = 0\cdot t + 9 = 9 \end{aligned}$$

for all t. So each point on the line also lies on the plane; i.e. the line lies within the plane. 2

 Name ______________

3. *(30 points)* The line L_1 is given by $x-3=y=1-z$, and the line L_2 is given by $x+1=y+4=z+7$.

a. Find a vector parallel to L_1. We know that $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ is parallel

[3]

to $\langle a,b,c\rangle$, so $\frac{x-3}{1}=\frac{y-0}{1}=\frac{z-1}{-1}$ is parallel to $\boxed{\langle 1,1,-1\rangle.}$

b. Find a vector parallel to L_2.

[3] As in a), $\frac{x-(-1)}{1}=\frac{y-(-4)}{1}=\frac{z-(-7)}{1}$ is parallel to $\boxed{\langle 1,1,1\rangle}$.

c. Find the point of intersection of L_1 and L_2.

[8] One way is to solve $\begin{cases} x-3=1-z \\ x+1=z+7 \end{cases}$ simultaneously. Adding the two equations gives 2 $2x-2=8$, so $2x=10$, so $x=5$ 2 and hence $z=-1$, 2 Then from $x-3=y$ we get $y=2$. 2 So the point is $\boxed{(5,2,-1).}$

d. Find an equation of the plane containing L_1 and L_2.

[16] The plane contains the point of intersection 2 $(5,2,-1)$.

Also ~~[illegible]~~ the plane is parallel 2 to $\langle 1,1,-1\rangle$ and $\langle 1,1,1\rangle$, so it is normal to their cross product, which is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}\,[1\cdot 1-(1\cdot -1)] - \vec{j}\,[1\cdot 1-1\cdot(-1)] + \vec{k}\,[1\cdot 1-1\cdot 1] =$$

2 2 2

$= 2\vec{i}-2\vec{j}+0\vec{k} = \langle 2,-2,0\rangle$. 4 So an equation for the plane 2 is $2(x-5)-2(y-2)+0(z-(-1))=0$, or ~~[illegible]~~ $2x-2y=6$, or $\boxed{x-y=3}$.

4. *(12 points)* For the curve $\mathbf{r}(t)=\langle t^2,\frac{1}{t},t^3\rangle$, find a unit tangent vector $\mathbf{T}(t)$ at the point (1,1,1).

A tangent vector at $(1,1,1)$, where $t=1$, 2

is $\vec{r}\,'(t)=\langle 2t,\frac{-1}{t^2},3t^2\rangle\Big|_{t=1}$ 4 $=\langle 2,-1,3\rangle$. 2

So a unit tangent vector is $\vec{T}(1)=\frac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|}=\frac{\langle 2,-1,3\rangle}{\sqrt{2^2+1^2+3^2}}=\boxed{\frac{\langle 2,-1,3\rangle}{\sqrt{14}}}.$

2 2

11

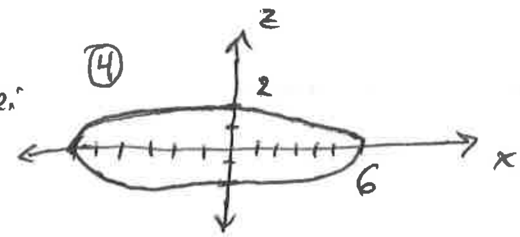
 Name ______________________

5. *(24 points)* For the surface $x^2 + 4y^2 + 9z^2 = 36$:

a. Sketch the trace (if any) in the xz-plane. Show the intercepts.

[6] When $y=0$, $x^2+9z^2=36$, (2) This is an ellipse:

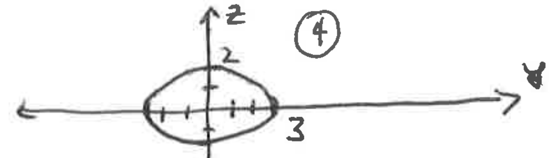
or $\left(\frac{x}{6}\right)^2+\left(\frac{z}{2}\right)^2=1$



[6] **b.** Sketch the trace (if any) in the yz-plane. Show the intercepts.

When $x=0$, (2) $4y^2+9z^2=36$, This is also an ellipse:

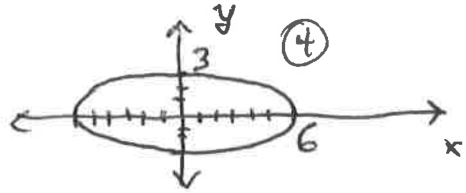
or $\left(\frac{y}{3}\right)^2+\left(\frac{z}{2}\right)^2=1$



[6] **c.** Sketch the trace (if any) in the xy-plane. Show the intercepts.

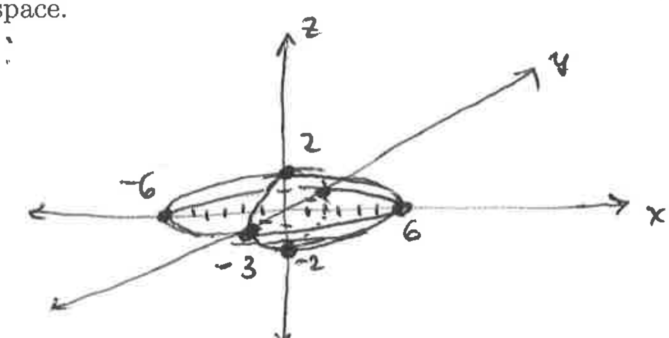
When $z=0$, $x^2+4y^2=36$, which is another ellipse:

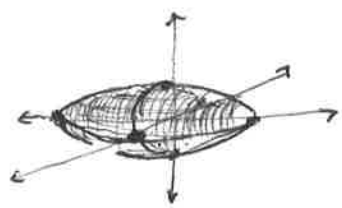
(2) or $\left(\frac{x}{6}\right)^2+\left(\frac{y}{3}\right)^2=1$



[6] **d.** Sketch the surface in space.

It is an ellipsoid:





6. *(12 points)* A curve is formed by the intersection of the surfaces $y = x^2$ and $z = x^2 + y^2$.

a. Find parametric equations for the curve.

Since a curve has one degree of freedom, we can just let one of the variables $x, y,$ or z be the free parameter t, and use the two equations to determine the other two variables.

Taking $x=t$, we then get $y=t^2$ and $z=t^2+(t^2)^2=t^2+t^4$.

(2) (2) So $\begin{cases} x=t \\ y=t^2 \\ z=t^2+t^4 \end{cases}$ (2)

b. Find a tangent vector to the curve at the point $(1, 1, 2)$.

At this point, we have $t=1$, (2)

and since $\vec{r}(t) = \langle t, t^2, t^2+t^4 \rangle$,

Then $\vec{r}\,'(t) = \langle 1, 2t, 2t+4t^3 \rangle$, (2)

and a tangent vector at the point is

$\vec{r}\,'(1) = \boxed{\langle 1, 2, 6 \rangle}$. (2)