EXAM 2
Math 2934
10-30-15

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(16 points)*

a) Suppose $f(x, y) = \ln(x^2 + y^2)$. Find the directional derivative of f at the point $P(2, 3)$ in the direction from P towards the point $Q(3, 5)$.

[redacted]
[10]

$$\frac{\partial f}{\partial x}\Big|_P = \frac{2x}{x^2+y^2}\Big|_P = \frac{2\cdot 2}{2^2+3^2} = \frac{4}{13} \text{ ②} ; \text{ and } \frac{\partial f}{\partial y}\Big|_P = \frac{2y}{x^2+y^2}\Big|_P = \frac{2\cdot 3}{2^2+3^2} = \frac{6}{13} \text{ ②}, \text{ so}$$

$\vec{\nabla} f|_P = \langle \frac{4}{13}, \frac{6}{13} \rangle$. The unit vector in the direction from P to Q is $\frac{\vec{v}}{|\vec{v}|}$, where $\vec{v} = \langle 3-2, 5-3 \rangle = \langle 1, 2 \rangle$.

So $\vec{u} = \frac{\langle 1,2 \rangle}{\sqrt{5}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$ ②, and $D_{\vec{u}} f|_P = \vec{\nabla} f|_P \cdot \vec{u} = \langle \frac{4}{13}, \frac{6}{13} \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$ ②

$$= \frac{4}{13\sqrt{5}} + \frac{12}{13\sqrt{5}} = \frac{16}{13\sqrt{5}} \text{ ②} = \boxed{\frac{16\sqrt{5}}{65}}$$

b) Find the maximum rate of change of f at the point $P(2, 3)$.

[redacted]
[3]

It is $|\vec{\nabla} f|_P| = |\langle \frac{4}{13}, \frac{6}{13} \rangle|$

$$= \sqrt{\left(\tfrac{4}{13}\right)^2 + \left(\tfrac{6}{13}\right)^2} = \sqrt{\frac{16+36}{13^2}} = \frac{\sqrt{52}}{\sqrt{13^2}} = \boxed{\frac{2\sqrt{13}}{13}}.$$

[redacted]
[3]

c) Give the direction in which the maximum rate of change of f at P occurs.

The direction of the gradient of f at P, which is $\boxed{\langle \frac{4}{13}, \frac{6}{13} \rangle}$.
(any vector parallel to $\langle \frac{4}{13}, \frac{6}{13} \rangle$ would also be a correct answer.)

2. *(15 points)* Suppose z is given as a function of x and y by the equation $x^2z^3 + y^3z = 51$.

[5] *a)* Find $\frac{\partial z}{\partial x}$, as a function of x, y, and z.

Letting $F(x,y,z) = x^2z^3 + y^3z - 51$, we have that z is defined by $F(x,y,z) = 0$, so

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \text{ ②} = \boxed{\frac{-(2xz^3)}{3x^2z^2 + y^3}} \text{ ③}$$

[5] *b)* Find $\frac{\partial z}{\partial y}$, as a function of x, y, and z.

As in part a),

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} \text{ ②} = \boxed{\frac{-3y^2z}{3x^2z^2 + y^3}} \text{ ③}.$$

[5] *c)* Find an equation for the tangent plane to the surface given by the equation at the point $(1, 2, 3)$. ②

At P, $\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \langle 2xz^3, 3y^2z, 3x^2z^2 + y^3 \rangle = \langle 54, 36, 35 \rangle$.

So an equation is $\boxed{54(x-1) + 36(y-2) + 35(z-3) = 0}$. ③

3. *(12 points)* A particle moves through space so that its position at time t is given by the parametric equations $x = \sin(2t)$, $y = \cos(2t)$, and $z = 5t$.

a) Find the velocity of the particle at the time when $t = \pi/4$. (The answer should be a vector.)

[6]

$$\vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \overset{②}{=} \langle \overset{①}{+2\cos(2t)}, \overset{①}{-2\sin(2t)}, \overset{①}{5} \rangle, \text{ so}$$

$$\text{At } t = \pi/4,\ \vec{v} = \langle 2\cos\tfrac{\pi}{2}, -2\sin\tfrac{\pi}{2}, 5 \rangle \overset{①}{=} \boxed{\langle 0, -2, 5 \rangle.}$$

b) Find the length of the path travelled by the particle between time $t = 0$ and time $t = 3$.

[6]

$$L = \int_0^3 |\vec{v}|\,dt \overset{②}{=} \int_0^3 \sqrt{\underset{①}{(2\cos(2t))^2} + (-2\sin 2t)^2 + 5^2}\,dt$$

$$= \int_0^3 \sqrt{4\underset{①}{[\cos^2(2t) + \sin^2(2t)]} + 25}\,dt = \int_0^3 \underset{②}{\sqrt{4+25}}\,dt = \int_0^3 \sqrt{29}\,dt = \underset{①}{\left[\sqrt{29}\,t\right]_0^3} = \boxed{3\sqrt{29}}$$

4. *(12 points)* Suppose $z = \sqrt{x^2 + 3y}$. Use differentials to estimate the value of z when $x = 1.06$ and $y = 1.08$.

$$z(x=1.06, y=1.08) = z(x=1, y=1) + \Delta z$$

$$= \sqrt{1^2 + 3\cdot y} + \Delta z = 2 + \Delta z, \text{ where}$$

$$\Delta z \cong \left(\frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}}\right)\Delta x \underset{②}{+} \left(\frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=1}}\right)\Delta y = \left(\underset{②}{\tfrac{1}{2}(x^2+3y)^{-\frac{1}{2}}\cdot 2x}\Big|_{\substack{x=1\\y=1}}\right)\Delta x + \left(\underset{②}{\tfrac{1}{2}(x^2+3y)^{-\frac{1}{2}}\cdot 3}\Big|_{\substack{x=1\\y=1}}\right)\Delta y$$

$$= \frac{1}{\sqrt{4}}\Delta x + \underset{②}{\frac{3}{2\sqrt{4}}\Delta y} = \frac{1}{2}(\Delta x) + \frac{3}{4}(\Delta y) = \underset{②}{\tfrac{1}{2}(0.06) + \tfrac{3}{4}(0.08)} = 0.03 + 0.06$$

$$\text{so } \Delta z \cong 0.09,$$

$$\text{and } \boxed{z \cong 2.09}\ ②$$

5. *(12 points)* Find the critical point of the function $f(x, y) = x^2 + 5y^2 + 2xy - x$.

Setting $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ equal to 0 gives ②

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + 2y - 1 = 0 \ ② \\ \dfrac{\partial f}{\partial y} = \underset{②}{10y + 2x} = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2y = 1 \ ② \\ y = \dfrac{-2}{10}x = \dfrac{-x}{5} \end{cases} \Rightarrow 2x - \frac{2x}{5} = 1$$

$$\Rightarrow \frac{8x}{5} = 1 \Rightarrow \overset{②}{x = \frac{5}{8}}, \text{ and } \overset{②}{y = \frac{-x}{5} = \frac{-1}{5}\cdot\frac{5}{8} = \frac{-1}{8}}.$$

So the critical point is $\boxed{(x, y) = \left(\frac{5}{8}, \frac{-1}{8}\right).}$

6. *(15 points)* Evaluate the double integral $\int_0^1 \int_2^3 (3x^2 + 2y)\, dy\, dx$.

~~$\int_0^1 \left[x^3 + 2yx \right]_{x=2}^{x=3}$~~ ② ②

$$\int_0^1 \left[3x^2 y + y^2 \right]_{y=2}^{y=3} dx =$$

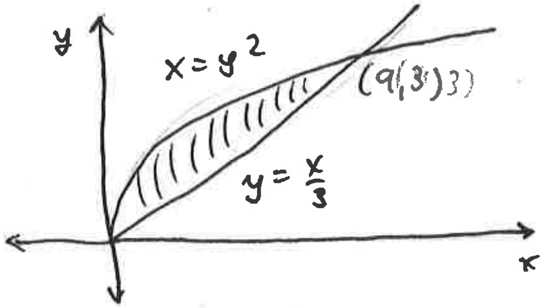
② ②

$$= \int_0^1 \left[(9x^2 + 9) - (6x^2 + 4) \right] dx =$$

$$= \int_0^1 (3x^2 + 5)\, dx = \left[x^3 + 5x \right]_0^1 = \boxed{6}$$

② ③ ②

7. *(18 points)* Evaluate $\iint_D 2y\, dA$, where D is the region between the line $y = x/3$ and the curve $x = y^2$ (see diagram).



$$\int_0^9 \int_{y=\frac{x}{3}}^{y=\sqrt{x}} 2y\, dy\, dx$$

② ② ② ②

$$= \int_0^9 \left[y^2 \right]_{y=\frac{x}{3}}^{y=\sqrt{x}} dx = \int_0^9 \left[\sqrt{x}^2 - \left(\frac{x}{3}\right)^2 \right] dx$$

② ②

$$= \int_0^9 \left[x - \frac{x^2}{9} \right] dx = \left[\frac{x^2}{2} - \frac{x^3}{3 \cdot 9} \right]_0^9$$

② ②

$$= \frac{9^2}{2} - \frac{9^3}{3 \cdot 9} = \frac{9^2}{2} - \frac{9^2}{3} = 81\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{81}{6} = \boxed{\frac{27}{2}}$$

②