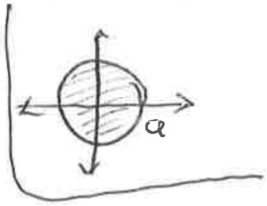
EXAM 3
Math 2934
12-4-15

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(15 points)* Evaluate $\iint_D \frac{1}{\sqrt{b^2+x^2+y^2}}\, dA$, where D is the disc in the xy-plane with center at the origin and radius a.



$$\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{b^2+r^2}}\, r\,dr\,d\theta = \int_0^{2\pi}\int_{b^2}^{b^2+a^2} \frac{1}{\sqrt{u}}\,\frac{1}{2}\,du\,d\theta \quad (1)$$

In polar coordinates,
$x^2+y^2=r^2$ (1) (2)
and $dA = r\,dr\,d\theta$, (1)
and $D = \{(r,\theta): 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$ (1)

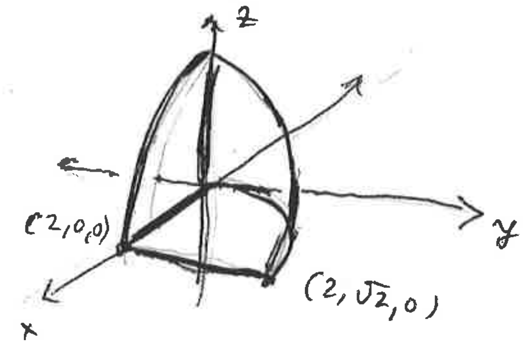
$u = b^2+r^2$ (1)
$du = 2r\,dr$ (1)
$r=0 \to u=b^2$ (1)
$r=a \to u=b^2+a^2$ (1)

$$= \frac{1}{2}\left(\int_0^{2\pi} d\theta\right)\left(\int_{b^2}^{b^2+a^2} u^{-\frac{1}{2}}\,du\right)$$

$$= \frac{1}{2}\cdot 2\pi \cdot (2) \left[\frac{u^{1/2}}{1/2}\right]_{b^2}^{b^2+a^2} (2)$$

$$= \boxed{2\pi\left[\sqrt{b^2+a^2}-b\right]} \quad (1)$$

2. *(20 points)* A solid E in the first octant is bounded by the planes $y=0$ and $z=0$ and by the surfaces $z=4-x^2$ and $x=y^2$ (see figure). Find $\iiint_E y\, dV$.



$$\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{x}}\int_{z=0}^{z=4-x^2} y\,dz\,dy\,dx = \quad (2)\ (2)\ (2)\ (2)$$

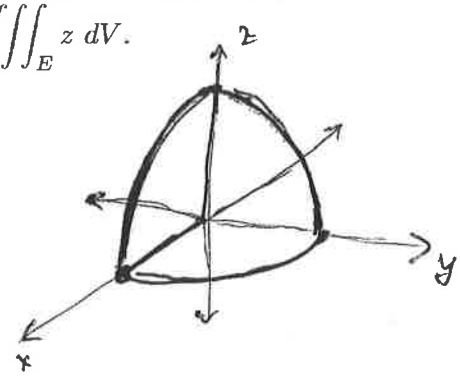
$$= \int_0^2\int_0^{\sqrt{x}} \Big[yz\Big]_{z=0}^{z=4-x^2} dy\,dx = \quad (2)$$

$$= \int_0^2\int_0^{\sqrt{x}} y(4-x^2)\,dy\,dx = \int_0^2 (4-x^2)\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \quad (2)\ (2)$$

$$= \int_0^2 (4-x^2)\frac{x}{2}\,dx = \frac{1}{2}\int_0^2 (4x-x^3)\,dx = \frac{1}{2}\left(2x^2-\frac{x^4}{4}\right)\Big]_0^2 \quad (2)\ (2)$$

$$= \frac{1}{2}\left(2\cdot 4 - \frac{16}{4}\right) = \boxed{2} \quad (2)$$

3. *(15 points)* A solid E in the first octant is bounded by the planes $x=0$, $y=0$, $z=0$ and the sphere of radius a (see figure). Use a triple integral in spherical coordinates to find $\iiint_E z\,dV$.



In spherical coordinates,

$E = \{ 0 \le \rho \le a,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2} \}$

and $z = \rho\cos\phi$, $dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$,

so the integral is

(1) (1)

$$\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho\cos\phi\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

(1) (1) (1)

$$= \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta = \left(\int_0^a \rho^3\, d\rho\right)\left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} \cos\phi\sin\phi\, d\phi\right)$$

$$\boxed{\begin{aligned} u &= \sin\phi \\ du &= \cos\phi\, d\phi \\ \phi = 0 &\to u = 0 \\ \phi = \tfrac{\pi}{2} &\to u = 1 \end{aligned}}$$

(2)

$$= \left[\frac{\rho^4}{4}\right]_0^a \cdot \frac{\pi}{2} \cdot \int_0^1 u\, du = \frac{a^4}{4}\cdot\frac{\pi}{2}\left[\frac{u^2}{2}\right]_0^1 = \boxed{\frac{\pi a^4}{16}}.$$

(2) (2) (2) (2)

4. *(15 points)* If

$$\mathbf{F}(x,y,z) = (y^2z + e^z)\mathbf{i} + (2xyz + z\cos y)\mathbf{j} + (xy^2 + \sin y + xe^z + 2z)\mathbf{k},$$

find a function f such that $\nabla f = \mathbf{F}$.

(1)

$f_x = y^2 z + e^z$ (1) $\Rightarrow f = xy^2z + xe^z + g(y,z) \Rightarrow$

(1) (1)

$\Rightarrow f_y = 2xyz + g_y \Rightarrow \cancel{2xyz} + g_y = \cancel{2xyz} + z\cos y \Rightarrow$

(1) (1)

$\Rightarrow g_y = z\cos y \Rightarrow g = z\sin y + h(z) \Rightarrow$

(1)

$\Rightarrow f = xy^2z + xe^z + z\sin y + h(z) \Rightarrow$

(1)

$\Rightarrow f_z = xy^2 + xe^z + \sin y + h'(z) \Rightarrow$

(1)

$\Rightarrow \cancel{xy^2} + \cancel{\sin y} + \cancel{xe^z} + h'(z) = \cancel{xy^2} + \cancel{\sin y} + \cancel{xe^z} + 2z \Rightarrow$

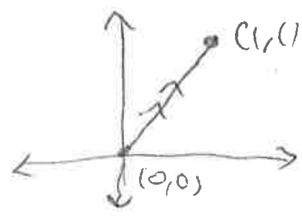
(1) (1)

$\Rightarrow h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow$

$\Rightarrow \boxed{f(x,y,z) = xy^2z + xe^z + z\sin y + z^2 + C}$ (4)

5. *(35 points)* Evaluate $\int_C 4xy\,dx + x\,dy$ where

a) C is the line segment from $(0,0)$ to $(1,1)$.



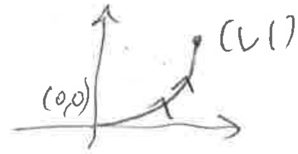
[8] Parametrize C as $\begin{cases} x = t \text{ (2)} \\ y = t \end{cases}$, $0 \le t \le 1$. (2)

Then $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 1$, so

$$\int_C 4xy\,dx + x\,dy = \int_0^1 4t\cdot t\,dt + t\,dt \text{ (2)} = \int_0^1 (4t^2 + t)\,dt$$

$$= \left[\frac{4t^3}{3} + \frac{t^2}{2}\right]_0^1 \text{ (1)} = \frac{4}{3} + \frac{1}{2} = \boxed{11/6} \text{ (1)}$$

b) C is the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.



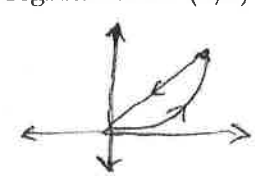
[8] Parametrize C as $\begin{cases} x = t \text{ (2)} \\ y = t^2 \text{ (2)} \end{cases}$, $0 \le t \le 1$.

Then $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$, so

$$\int_C 4xy\,dx + x\,dy = \int_0^1 4t\cdot t^2\,dt + t\cdot 2t\,dt \text{ (2)} = \int_0^1 (4t^3 + 2t^2)\,dt =$$

$$= \left[t^4 + \frac{2t^3}{3}\right]_0^1 \text{ (1)} = 1 + \frac{2}{3} = \boxed{5/3} \text{ (1)}$$

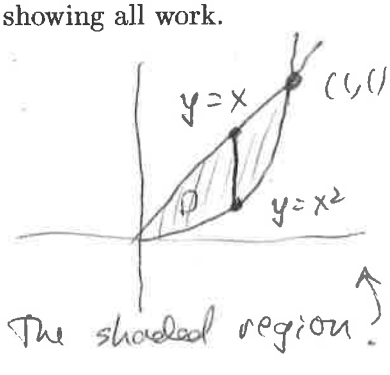
c) C goes along the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and then back along the line segment from $(1,1)$ to $(0,0)$. (Use your answers to parts a) and b).)



[4] It's the answer to part b) minus the answer to part a),

So $\int_C 4xy\,dx + x\,dy = \frac{5}{3} - \frac{11}{6} = \boxed{-1/6}$. (4)

d) Use Green's Theorem to rewrite the integral over the closed curve C in part c) as a double integral. Verify Green's Theorem by evaluating the double integral as an iterated integral, showing all work.



[15] Using $\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$, (2)

with $P = 4xy$, $Q = x$,

$\frac{\partial P}{\partial y} = 4x$, $\frac{\partial Q}{\partial x} = 1$, we have (2)

$\oint_C 4xy\,dx + x\,dy = \iint_D (1 - 4x)\,dA$ where D is the shaded region.

To evaluate the double integral, write it as

$$\text{(2)} \int_0^1 \text{(2)} \int_{x^2}^{x} (1-4x)\,dy\,dx = \int_0^1 \left[y - 4xy\right]_{y=x^2}^{y=x} \text{ (2)}\,dx =$$

$$= \int_0^1 \left[(x - 4x^2) - (x^2 - 4x^3)\right]dx \text{ (2)} = \int_0^1 (x - 5x^2 + 4x^3)\,dx =$$

$$= \left[\frac{x^2}{2} - \frac{5x^3}{3} + x^4\right]_0^1 \text{ (2)} = \frac{1}{2} - \frac{5}{3} + 1 = \frac{3}{6} - \frac{10}{6} + \frac{6}{6} \text{ (1)} = \boxed{\frac{-1}{6}},$$ agreeing with part c).