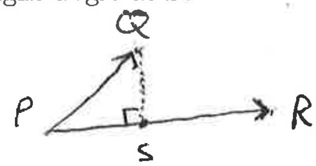
EXAM 1 Name *key*
Math 2934
5-22-15

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* In the diagram, $\overrightarrow{PQ} = \langle 3, 5\rangle$ and $\overrightarrow{PR} = \langle 7, 1\rangle$. Triangle PQS has a right angle at S.



[5] **a.** Find $\overrightarrow{QR}$. (2) $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$, so

$\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ} = \langle 7,1\rangle - \langle 3,5\rangle = \boxed{\langle 4,-4\rangle}$. (3)

[5] **b.** Find the cosine of the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}|\,|\overrightarrow{PR}|} \ (2) = \frac{3\cdot 7 + 5\cdot 1}{\sqrt{3^2+5^2}\ \sqrt{7^2+1^2}} = \boxed{\frac{26}{\sqrt{34}\sqrt{50}}} \ (3)$$

[5] **c.** Find $|\overrightarrow{PS}|$. (2)

$$|\overrightarrow{PS}| = |\overrightarrow{PQ}|\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PR}|} = \boxed{\frac{26}{\sqrt{50}}} \ (3)$$

[5] **d.** Find $\overrightarrow{PS}$.

A unit vector in the direction of $\overrightarrow{PS}$ is $\dfrac{\overrightarrow{PR}}{|\overrightarrow{PR}|} = \dfrac{\langle 7,1\rangle}{\sqrt{50}}$. (2)

$$\text{So } \overrightarrow{PS} = |\overrightarrow{PS}|\cdot\frac{\langle 7,1\rangle}{\sqrt{50}} \ (2) = \frac{26}{\sqrt{50}}\cdot\frac{\langle 7,1\rangle}{\sqrt{50}} = \frac{26}{50}\langle 7,1\rangle \ (1)$$
$$= \frac{13}{25}\langle 7,1\rangle$$
$$= \boxed{\left\langle \frac{91}{25}, \frac{13}{25}\right\rangle}.$$

2. *(8 points)* The line L has vector equation $\mathbf{r}(t) = \langle 1,4,0\rangle + t\langle 2,1,3\rangle$. Is this line parallel to the plane with equation $x - 5y + z = 4$? (Justify your answer!)

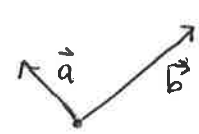
(2) – From the equation of the plane we see that a vector orthogonal to the plane is $\vec{n} = \langle 1,-5,1\rangle$.

(2) – A vector is parallel to the plane if and only if it is orthogonal to $\vec{n}$.

(2) – Since $\langle 2,1,3\rangle\cdot\langle 1,-5,1\rangle = 2-5+3 = 0$, then $\langle 2,1,3\rangle \perp \vec{n}$, so $\langle 2,1,3\rangle$ is parallel to the plane.

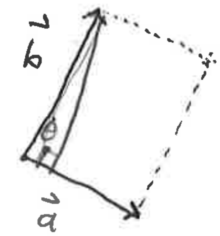
(2) – Since L is parallel to $\langle 2,1,3\rangle$, then L is also parallel to the plane.

3. *(6 points)* The two vectors $\mathbf{a}$ and $\mathbf{b}$ are positioned as shown in the diagram on this piece of paper as it lies horizontally on your desk. Does $\mathbf{a}\times\mathbf{b}$ point upwards or downwards? (Briefly explain your answer.)



(4) If we put the thumb of our right hand in the direction of $\vec{a}$, and the index finger towards $\vec{b}$, then the palm faces downwards. So by the right-hand rule, $\vec{a}\times\vec{b}$ points downwards. (2)

4. *(10 points)* Find the area of the parallelogram in the diagram, if the vectors shown are given by $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$.



The area of the parallelogram is the magnitude of $|\vec{a} \times \vec{b}|$. (i.e. $|\vec{a}||\vec{b}|\sin\theta$) 2

We have $\vec{a} = \langle 1, -2, 0 \rangle$ and $\vec{b} = \langle 0, 1, 3 \rangle$, so 6

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}(-2\cdot 3 - 1\cdot 0) - \vec{j}(1\cdot 3 - 0\cdot 0) + \vec{k}(1\cdot 1 - 0\cdot -2)$$

or $\vec{a} \times \vec{b} = -6\vec{i} - 3\vec{j} + \vec{k} = \langle -6, -3, 1 \rangle$. 2

So $|\vec{a} \times \vec{b}| = \sqrt{(-6)^2 + (-3)^2 + 1^2} = \sqrt{36+9+1} = \boxed{\sqrt{46}.}$

5. *(16 points)* Find an equation of the plane containing the points $P(1, 0, 0)$, $Q(1, 1, 1)$, and $R(2, 0, 5)$. Show all work.

Two vectors parallel to the plane are 2

$\vec{PQ} = \langle 1-1, 1-0, 1-0 \rangle = \langle 0, 1, 1 \rangle$ 2 and $\vec{PR} = \langle 2-1, 0-0, 5-0 \rangle = \langle 1, 0, 5 \rangle$.

So a vector normal to the plane is 6

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 0 & 5 \end{vmatrix} = \vec{i}(5-0) - \vec{j}(0-1) + \vec{k}(0-1)$$

2

$$= 5\vec{i} + \vec{j} - \vec{k} = \langle 5, 1, -1 \rangle.$$

Since $P(1,0,0)$ is on the plane, an equation for the plane is therefore

$5(x-1) + 1(y-0) + (-1)(z-0) = 0$, or $\boxed{5x + y - z = 5}$.

4

6. *(12 points)* A line is given by the symmetric equations $x = y - 2 = \frac{z-1}{2}$. Find the coordinates of the point where the line intersects the plane $x + y + z = 11$.

→ One way to do this is to first write parametric equations for the line, and then find the value of t where the line intersects the plane.

→ Parametric equations for the line are $\begin{cases} x = t \\ y - 2 = t \\ \frac{z-1}{2} = t \end{cases}$ or $\begin{cases} x = t \\ y = 2 + t \\ z = 1 + 2t \end{cases}$. 3

→ The line intersects the plane when

$11 = x + y + z = (t) + (2+t) + (1+2t)$ 3, or $11 = 3 + 4t$,

so $t = 2$. 3

→ At that value of t, $x = 2$, $y = 4$, $z = 5$ 3. So the point is $(2, 4, 5)$.

7. *(12 points)* Draw the trace of the surface $z = x^2 + 4y^2 - 1$ in each of the following planes. Mark the points on the x and y axes where the traces cross them.

a. In the plane $z = 0$.

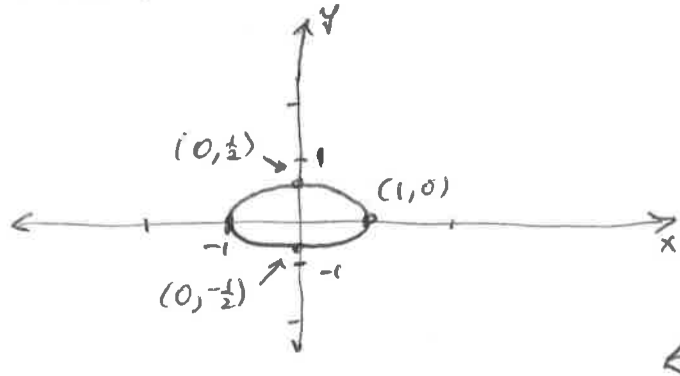
[4] $0 = x^2 + 4y^2 - 1$

$\Downarrow$

$x^2 + 4y^2 = 1$

$\Downarrow$

$x^2 + \frac{y^2}{(\frac{1}{2})^2} = 1$

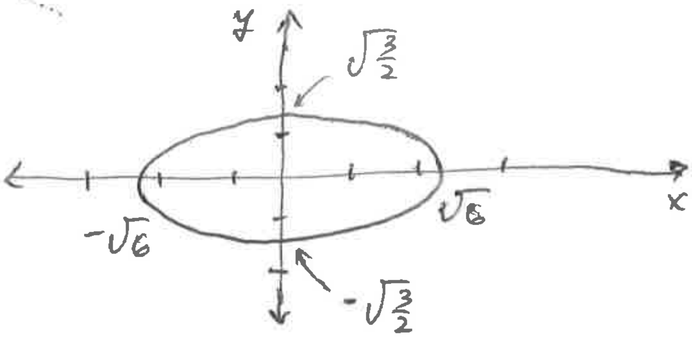


[4] **b.** In the plane $z = 5$. 3

$5 = x^2 + 4y^2 - 1$

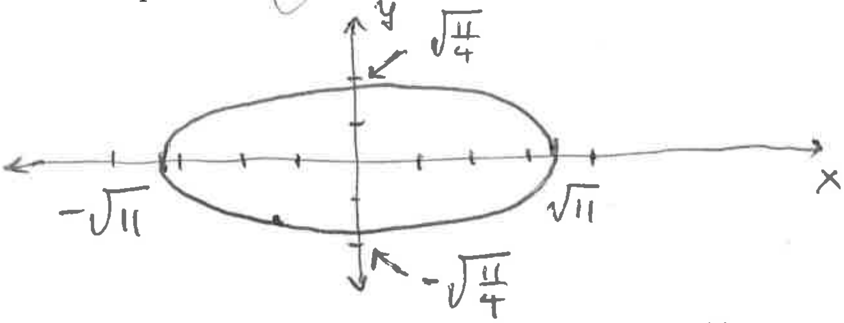
$x^2 + 4y^2 = 6$

$\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{(\sqrt{\frac{3}{2}})^2}$



I messed up a little bit here. I wanted the numbers to come out to be simple, so I should have chosen $z = 3$ and $z = 8$ instead.

[4] **c.** In the plane $z = 10$. 8



$10 = x^2 + 4y^2 - 1$

$\Downarrow$

$11 = x^2 + 4y^2$

$\Downarrow$

$1 = \frac{x^2}{(\sqrt{11})^2} + \frac{y^2}{(\sqrt{\frac{11}{4}})^2}$

8. *(16 points)* A curve is given by the vector equation $\mathbf{r}(t) = \cos t\ \mathbf{i} + 3t\ \mathbf{j} + \sin t\ \mathbf{k}$.

a. Find a *unit* vector tangent to the curve at the point $(1, 0, 0)$.

[10] A tangent vector to the curve at $(1,0,0)$, which is where $t = 0$, is (2)

$\vec{r}\,'(0) = \langle -\sin t, 3, \cos t \rangle \big|_{t=0} = \langle 0, 3, 1 \rangle.$ (2) (2) (2)

A unit tangent vector to the curve at $(1,0,0)$ is $\frac{\langle 0,3,1 \rangle}{\sqrt{0^2+3^2+1^2}} = \boxed{\frac{1}{\sqrt{10}} \langle 0,3,1 \rangle.}$ (2)

b. Find parametric equations for the line tangent to the curve at the point $(1, 0, 0)$.

[6] The line is parallel to $\langle 0,3,1 \rangle$, so it has (2) vector equation $\vec{r}(t) = \langle 1,0,0 \rangle + t\langle 0,3,1 \rangle$, and parametric equations

$x = 1 + 0 \cdot t$
$y = 0 + 3t$
$z = 0 + 1t$

or

$\boxed{\begin{array}{l} x = 1 \\ y = 3t \\ z = t \end{array}}$ (4)