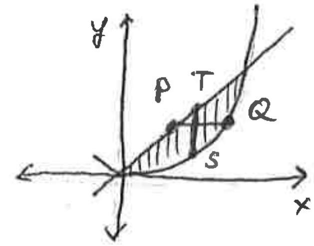
EXAM 3
Math 2934
6-18-15

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(15 points)* The region D in the plane is bounded by the curves $y = x^2$ and $y = x$ (see figure).



a) Evaluate $\iint_D x\, dx\, dy$.

On $y=x^2$, we have $x=\sqrt{y}$
At P, $x=y$
At Q, $x=\sqrt{y}$.

$$\int_0^1 \int_y^{\sqrt{y}} x \, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_{x=y}^{x=\sqrt{y}} dy = \int_0^1 \left[\frac{y}{2} - \frac{y^2}{2}\right] dy$$

$$= \left[\frac{y^2}{4} - \frac{y^3}{6}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \boxed{\frac{1}{12}.}$$

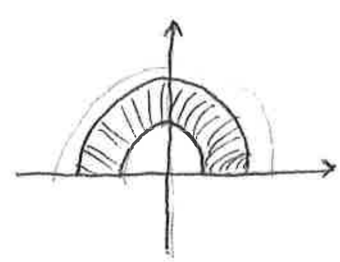
b) Evaluate $\iint_D x\, dy\, dx$.

At S, we have $y=x^2$.
At T, we have $y=x$.

$$\int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 \left[xy\right]_{y=x^2}^{y=x} dx = \int_0^1 \left[x^2 - x^3\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}.}$$

2. *(15 points)* Let D be the region lying above the x-axis and between the circles of radius 1 and 2 centered at the origin (see figure). Evaluate $\iint_D y\, dA$.



In polar coordinates, D is the region where $1 \le r \le 2$ and $0 \le \theta \le \pi$. Also, $y = r\sin\theta$.

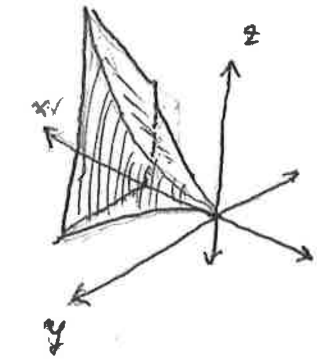
So

$$\iint_D y\, dA = \int_0^{\pi} \int_1^2 r\sin\theta \; r\, dr\, d\theta$$

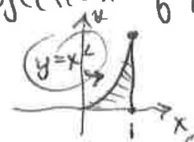
$$= \int_0^{\pi}\int_1^2 r^2 \sin\theta\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^3}{3}\sin\theta\right]_{r=1}^{r=2} d\theta = \int_0^{\pi} \left(\frac{8}{3} - \frac{1}{3}\right)\sin\theta\, d\theta$$

$$= \frac{7}{3}\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} = \frac{7}{3}\left[-(-1) - (-1)\right] = \boxed{\frac{14}{3}.}$$

3. *(15 points)* Let E be the region bounded on the sides by the surface $y = x^2$ and the planes $x = 1$ and $y = 0$, below by the plane $z = 0$, and above by the surface $z = x^2 + 2y$. Evaluate $\iiint_E xy\, dV$.



The projection of E onto the xy-plane is the region:



So the triple integral is

$$\int_0^1 \int_0^{x^2} \int_0^{x^2+2y} xy\, dz\, dy\, dx =$$

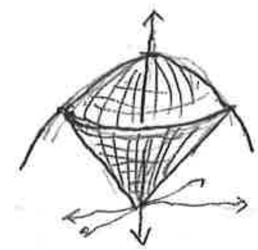
$$\int_0^1 \int_0^{x^2} \left[xyz\right]_{z=0}^{z=x^2+2y} dy\, dx =$$

$$\int_0^1 \int_0^{x^2} xy(x^2+2y)\, dy\, dx = \int_0^1 \int_0^{x^2} (x^3y + 2xy^2)\, dy\, dx =$$

$$= \int_0^1 \left[\frac{x^3y^2}{2} + \frac{2xy^3}{3}\right]_{y=0}^{y=x^2} dx = \int_0^1 \left[\frac{x^3 \cdot x^4}{2} + \frac{2x \cdot x^6}{3}\right] dx =$$

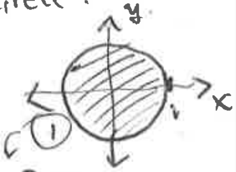
$$= \int_0^1 \left(\frac{1}{2} + \frac{2}{3}\right) x^7\, dx = \left[\frac{7}{6}\frac{x^8}{8}\right]_0^1 = \boxed{\frac{7}{48}}$$

4. *(20 points)* Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the cone $z^2 = x^2 + y^2$ (see figure). Use a triple integral in cylindrical coordinates.



The cone is given by $z^2 = r^2$, or $z = r$, and the sphere by $r^2 + z^2 = 2$, or $z = \sqrt{2-r^2}$.

The intersection of the cone and sphere is where $x^2+y^2+z^2 = z^2+z^2 = 2$, or $z = 1$, so $r = 1$. The projection of E on the xy-plane is the circle:



So

$$V = \iiint_E 1 \cdot dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} 1\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[rz\right]_{z=r}^{z=\sqrt{2-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[r\sqrt{2-r^2} - r^2\right] dr\, d\theta$$

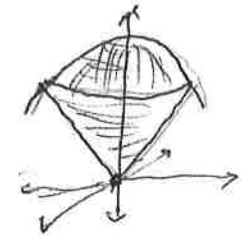
$$= 2\pi \left(\int_0^1 r\sqrt{2-r^2}\, dr - \int_0^1 r^2\, dr\right).$$

Put $u = 2-r^2$, $du = -2r\,dr$, $\frac{du}{-2} = r\,dr$; $r=0 \to u=2$, $r=1 \to u=1$

$$\to = 2\pi \left[\int_2^1 \sqrt{u}\, \frac{du}{-2} - \left[\frac{r^3}{3}\right]_0^1\right] = 2\pi \left[-\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right]_2^1 - \frac{2\pi}{3}$$

$$= 2\pi\left[-\frac{1}{3} + \frac{2^{3/2}}{3} - \frac{1}{3}\right] = \boxed{2\pi\left[\frac{2^{3/2}-2}{3}\right]}$$

5. *(20 points)* Let E be the region bounded above by the sphere $\rho = 1$ and below by the cone $\phi = \pi/4$ (see figure). Find $\iiint_E z\, dV$, using an integral in spherical coordinates.



The region is given in spherical coords. by $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le 1$. Also, $z = \rho\cos\phi$. So

$$\iiint_E z\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 (\rho\cos\phi)(\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta)$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos\phi\sin\phi \left[\frac{\rho^4}{4}\right]_0^1 d\phi\, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\cos\phi\sin\phi}{4}\, d\phi\, d\theta =$$

$$\boxed{\begin{array}{ll} u = \sin\phi & \phi = 0 \to u = 0 \\ du = \cos\phi\, d\phi & \phi = \frac{\pi}{4} \to u = \frac{1}{\sqrt{2}} \end{array}}$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \frac{u}{4}\, du\, d\theta = \int_0^{2\pi} \left[\frac{u^2}{8}\right]_0^{\frac{1}{\sqrt{2}}} d\theta = \int_0^{2\pi} \frac{1}{16}\, d\theta = \frac{2\pi}{16} = \boxed{\frac{\pi}{8}}.$$

6. *(15 points)* Evaluate $\int_C x^2y\, dx + x\, dy + z\, dz$, where C is the curve given by $x = t$, $y = t^2$, and $z = t^3$ from $t = 0$ to $t = 1$.

On C, we have $\begin{cases} dx = 1\, dt \\ dy = 2t\, dt \\ dz = 3t^2\, dt \end{cases}$, so

$$\int_C x^2y\, dx + x\, dy + z\, dz = \int_0^1 t^2 \cdot t^2 \cdot 1\, dt + t \cdot 2t\, dt + t^3 \cdot 3t^2\, dt$$

$$= \int_0^1 (t^4 + 2t^2 + 3t^5)\, dt$$

$$= \left[\frac{t^5}{5} + \frac{2t^3}{3} + \frac{3t^6}{6}\right]_0^1 = \frac{1}{5} + \frac{2}{3} + \frac{1}{2}$$

$$= \frac{6}{30} + \frac{20}{30} + \frac{15}{30} = \boxed{\frac{41}{30}}.$$