Quiz 7

Name: key

1. Suppose $\mathbf{F}$ is the vector field in $\mathbf{R}^3$ given by

$$\mathbf{F} = (2xy^3z + ye^{xy})\ \mathbf{i} + (3x^2y^2z + xe^{xy} + 2y)\ \mathbf{j} + (x^2y^3 - \sin z)\ \mathbf{k},$$

and C is the curve parameterized by $x = \cos t$, $y = \sin t$, and $z = t$ for $0 \leq t \leq 2\pi$.

[6] **a.** Find a potential function for $\mathbf{F}$.

$$\frac{\partial f}{\partial x} = 2xy^3z + ye^{xy} \Rightarrow f = \int^x (2xy^3z + ye^{xy}) = x^2y^3z + e^{xy} + g(y,z)$$

$$\frac{\partial f}{\partial y} = 3x^2y^2z + xe^{xy} + 2y \Rightarrow 3x^2y^2z + xe^{xy} + \frac{\partial g}{\partial y} = 3x^2y^2z + xe^{xy} + 2y$$

$$\Rightarrow \frac{\partial g}{\partial y} = 2y \Rightarrow g = y^2 + h(z) \Rightarrow f = x^2y^3z + e^{xy} + y^2 + h(z)$$

$$\frac{\partial f}{\partial z} = x^2y^3 - \sin z \Rightarrow x^2y^3 + h'(z) = x^2y^3 - \sin z \Rightarrow h'(z) = -\sin z \Rightarrow h(z) = \cos z + C$$

so $\boxed{f = x^2y^3z + e^{xy} + y^2 + \cos z + C}$

[4] **b.** Use your answer to part **a** to find the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\int_C \vec{F} \cdot d\vec{r} = f\Big|_{t=2\pi} - f\Big|_{t=0} = f(x(2\pi), y(2\pi), z(2\pi)) - f(x(0), y(0), z(0)) =$$

$$= f(1, 0, 2\pi) - f(1, 0, 0) = (0 + e^0 + 0 + 1) - (0 + e^0 + 0 + 1) = \boxed{0}$$

2. Suppose $\mathbf{F}$ is the vector field in $\mathbf{R}^2$ given by

$$\mathbf{F}(x, y) = P\ \mathbf{i} + Q\ \mathbf{j} = y\ \mathbf{i} - x\ \mathbf{j}$$

and D is the unit circle (the circle with center at the origin and radius 1), with boundary C.

[6] **a.** Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, using the parametrization $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \text{ so } \int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=2\pi} y\,dx - x\,dy = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt$$

$$= \int_0^{2\pi} (-1)\,dt = \boxed{-2\pi}$$

[4] **b.** Compute $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$.

$$= \iint_D \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\right) dA = \iint_D (-1 - 1)\,dA = -2 \iint_D dA = -2(\text{area of } D) = \boxed{-2\pi}$$