EXAM 1
Math 3113
5-27-16

Name key

There are 6 problems totalling 100 points. Please show all work required to reach your answers. You may write on the back sides of the pages if you need more room.

1. *(12 points)* Substitute $y = ax^3$ into the differential equation $x^5 y'' = y^2$ to determine all values of the constant a for which $y = ax^3$ is a solution of the equation.

If $y = ax^3$, then $y' = 3ax^2$ (2) and $y'' = 6ax$. (2)

Also $y = ax^3 \Rightarrow y^2 = (ax^3)^2 = a^2x^6$. (2) So

$x^5 y'' = y^2 \Leftrightarrow x^5 \cdot 6ax = a^2x^6$ (2) $\Leftrightarrow 6ax^6 = a^2x^6 \Leftrightarrow$

$6a = a^2 \Leftrightarrow a^2 - 6a = 0$ (2) $\Leftrightarrow a(a-6) = 0 \Leftrightarrow \boxed{a = 0 \text{ or } a = 6}$. (2)

The only solutions of this form are $y = 0$ and $y = 6x^3$.

2. *(20 points)* Find the solution of $\dfrac{dy}{dx} = \dfrac{3y - xy}{x}$ which satisfies $y(1) = 3e^{-1}$. Give y explicitly as a function of x.

There are two ways to look at this equation; either as a linear equation or as a separable equation.

Separable: (2) $\dfrac{dy}{dx} = \dfrac{3y - xy}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{(3-x)\cdot y}{x}$ (2) $\Rightarrow \displaystyle\int \frac{1}{y}\,dy = \int \frac{3-x}{x}\,dx$ (2) (2) (2)

$\Rightarrow \ln|y| = \int \frac{3}{x} - 1\,dx \Leftrightarrow \ln|y| = 3\ln|x| - x + C \Leftrightarrow |y| = e^{3\ln|x| - x + C} \Leftrightarrow$

$\Leftrightarrow |y| = |x|^3 \cdot e^{-x} \cdot e^C$ (2) $\Leftrightarrow y = \pm e^C e^{-x}|x|^3 \Leftrightarrow y = D|x|^3 e^{-x}$.

Since we are looking for a solution near $x = 1$, we can assume $x > 0$, so $|x| = x$, so $y = Dx^3e^{-x}$. (2) Since $y = 3e^{-1}$ when $x = 1$, then $3e^{-1} = D \cdot e^{-1}$, (2)

so $D = 3$, so $\boxed{y = 3x^3e^{-x}}$. (2)

Linear: $\dfrac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = \dfrac{x-3}{x}$ (2) and $Q(x) = 0$. (2) Then $\int P(x)\,dx =$ (2)

$= \int 1 - \frac{3}{x}\,dx = x - 3\ln x$, so $e^{\int P(x)dx} = e^{x - 3\ln x} = e^x \cdot x^{-3}$ (2)

Second solution to #2, continued

Multiplying both sides of $\frac{dy}{dx} + \frac{x-3}{x} y = 0$

by the integrating factor $e^{\int \frac{x-3}{x} dx}$ gives

$$e^{\int \frac{x-3}{x} dx} \cdot \frac{dy}{dx} + \left(\frac{x-3}{x}\right) e^{\int \frac{x-3}{x} dx} \cdot y = 0$$

$$\frac{d}{dx}\left(e^{\int \frac{x-3}{x} dx} \cdot y \right) = 0$$

$$e^{\int \frac{x-3}{x} dx} \cdot y = C \quad 2$$

As we saw in the other solution, $e^{\int \frac{x-3}{x} dx} = D e^{x} \cdot x^{-3}$. We can absorb the constant D into the constant C on the other side

$$e^{x} \cdot x^{-3} y = C$$

$$y = x^3 e^{-x} C \quad 2$$

2

Since $y = 3e^{-1}$ when $x = 1$, then $3e^{-1} = e^{-1} \cdot C$, so $C = 3$

2

and $\boxed{y = 3x^3 e^{-x}.}$ 2

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3. *(13 points)* Find the solution of $\frac{dy}{dx} = 3x^2 + \frac{5}{x^2}$ which satisfies $y(1) = 4$. (2) (2)

(2)

$$y = \int 3x^2 + \frac{5}{x^2}\,dx = \int 3x^2 dx + \int 5x^{-2} dx = x^3 + \frac{5x^{-1}}{-1} + C,$$

(2) (2)

or $y = x^3 - \frac{5}{x} + C$. Since $y=4$ when $x=1$, then $4 = 1 - 5 + C$, so

$C = 8$, so $\boxed{y = x^3 - \frac{5}{x} + 8}$. (1)

(2)

4. *(20 points)* For the equation

$$\frac{dy}{dx} + 4xy = x + e^{-2x^2},$$

[15] a) Find the general solution, giving y explicitly as a function of x.

(2)

This is a linear equation, with integrating factor $e^{\int 4x\,dx} = e^{2x^2}$.

(2)

Multiplying the equation by e^{2x^2} gives

$$e^{2x^2}\frac{dy}{dx} + 4xe^{2x^2}y = xe^{2x^2} + 1,$$

so

$$\frac{d}{dx}\left(e^{2x^2}\cdot y\right) = xe^{2x^2} + 1, \text{ (1) so}$$

$$e^{2x^2}\cdot y = \int (xe^{2x^2} + 1)\,dx = \left(\int xe^{2x^2}dx\right) + x + C = \frac{1}{4}\int e^u du + x + C$$

(2) ~~abc~~ (2)

$u = 2x^2$

$du = 4x\,dx$ (2)

$$= \frac{1}{4}e^u + x + C$$

$$= \frac{1}{4}e^{2x^2} + x + C. \text{ (2)}$$

So $e^{2x^2}y = \frac{1}{4}e^{2x^2} + x + C$, so $y = e^{-2x^2}\left(\frac{1}{4}e^{2x^2} + x + C\right)$ (1)

$$\boxed{y = \frac{1}{4}\cdot 1 + xe^{-2x^2} + Ce^{-2x^2}}$$

[5] b) Find the particular solution satisfying $y(0) = 1$.

Since $y = 1$ when $x = 0$, then

(1) $1 = \frac{1}{4} + 0 + C\cdot 1$, so (1) (1)

$C = \frac{3}{4}$, (1) so

$$\boxed{y = \frac{1}{4} + xe^{-2x^2} + \frac{3}{4}e^{-2x^2}} \text{ (1)}$$

5. *(15 points)* Bacteria are growing in a culture according to the natural population growth equation. At time $t = 3$ there are 10,000 bacteria present, and at time $t = 6$ there are 50,000 bacteria present. How many bacteria were present at time $t = 0$? (Remember to show work justifying your answer.)

$P = De^{kt}$ 3 and $P = 10{,}000$ at $t=3 \Rightarrow 10{,}000 = De^{3k}$ 2

$P = 50{,}000$ at $t=6 \Rightarrow 50{,}000 = De^{6k}$ 2

Solving the first equation for D gives $D \overset{2}{=} \dfrac{10{,}000}{e^{3k}}$, so from the second equation we get $50{,}000 = \dfrac{10{,}000}{e^{3k}} \cdot e^{6k}$, or $5 = e^{6k-3k}$ 2 $\Rightarrow 5 = e^{3k} \Rightarrow$

$k = \dfrac{\ln 5}{3}$ 1. So $D = \dfrac{10{,}000}{e^{3\left(\frac{\ln 5}{3}\right)}} = \dfrac{10{,}000}{e^{\ln 5}} = \dfrac{10{,}000}{5} \overset{2}{=} 2{,}000$. At time $t=0$, 1

$P = De^0 = D = \boxed{2{,}000}$.

6. *(20 points)* Hot water is set out to cool in a room where the room temperature is 25° C. At time $t = 0$, the temperature of the water is 100° C, and at time $t = 10$ minutes the temperature of the water is 80° C. At what time t does the temperature of the water reach 50° C? (Your answer may involve logarithms which you do not have to evaluate.)

$\dfrac{dT}{dt} = k(T-A)$ 1, where here $A = 25$ 1, so

$\displaystyle\int \frac{dT}{T-25} = \int k\,dt$ 1 $\Rightarrow \ln(T-25) = kt + C \Rightarrow T - 25 = e^{kt+C} = De^{kt}$ 1

$\Rightarrow T = 25 + De^{kt}$. 1

Since $T = 100$ when $t=0$, then $100 = 25 + De^0$ 2

$\Rightarrow 75 = D$ 1

$\Rightarrow T = 25 + 75e^{kt}$ 1

Since $T=80$ when $t=10$, then $80 = 25 + 75e^{10k}$, 2

so $55 = 75e^{10k} \Rightarrow k = \frac{1}{10}\ln\left(\frac{55}{75}\right)$ 2

$\Rightarrow T = 25 + 75e^{\left(\frac{1}{10}\ln\frac{55}{75}\right)\cdot t}$ 1

When $T=50$, then $50 = 25 + 75e^{\left(\frac{1}{10}\ln\left(\frac{55}{75}\right)\right)t}$ 2

$\Rightarrow \underbrace{\frac{25}{75}}_{=\frac{1}{3}} = e^{\left(\frac{1}{10}\ln\left(\frac{55}{75}\right)\right)t} \Rightarrow \ln\left(\frac{1}{3}\right) = \frac{1}{10}\ln\left(\frac{55}{75}\right)t$ 2

$\Rightarrow$ ~~[illegible]~~ 2 $\boxed{t = \dfrac{10\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{11}{15}\right)}}$