EXAM 2
Math 3113
6-10-16

Name key

There are 6 problems totalling 100 points. Please show all work required to reach your answers. You may write on the back sides of the pages if you need more room.

1. *(15 points)* For the exact equation

$$(e^y + y\cos x + x) + (\sin x - \sin y + xe^y)y' = 0,$$

find the general solution (you can give the solution in implicit form).

$F = \int^x (e^y + y\cos x + x) = xe^y + y\sin x + \frac{x^2}{2} + g(y)$; so.

$\frac{\partial F}{\partial y} = xe^y + \sin x + g'(y)$. Since this must equal $\sin x - \sin y + xe^y$,

then $g'(y) = -\sin y$, so $g(y) = \cos y$. Therefore

$$F = xe^y + y\sin x + \frac{x^2}{2} + \cos y$$

satisfies $\frac{\partial F}{\partial x} = e^y + y\cos x + x$ and $\frac{\partial F}{\partial y} = \sin x - \sin y + xe^y$.

The general solution is given implicitly by the equation $F = \text{constant}$,

or $\boxed{xe^y + y\sin x + \frac{x^2}{2} + \cos y = C}$, where C is an arbitrary constant.

2. *(15 points)* Find the general solution of the reducible second-order equation $xy'' - y' = 3x^2$.

Let $p = y'$; then $y'' = p'$ and the equation becomes

$xp' - p = 3x^2$, or $p' + (-\frac{1}{x})\cdot p = 3x$. This is a linear first-order

differential equation for p, with integrating factor $e^{\int -\frac{1}{x}dx} =$

$= e^{-\ln x} = \frac{1}{(e^{\ln x})} = \frac{1}{x}$. Multiplying both sides by $\frac{1}{x}$ gives

$\frac{1}{x}\cdot p' + (\frac{-1}{x^2})\cdot p = 3$, or $(\frac{1}{x}\cdot p)' = 3$, so $\frac{1}{x}\cdot p = 3x + C$,

and $p = 3x^2 + Cx$. Thus $\frac{dy}{dx} = 3x^2 + Cx$, so

$\boxed{y =} \int (3x^2 + Cx)\,dx = \boxed{x^3 + \frac{C x^2}{2} + D}$ (or by letting $E = \frac{C}{2}$,

$\boxed{y = x^3 + Ex^2 + D}$.)

3. *(10 points)* For the three functions $y_1 = e^x$, $y_2 = e^{-x}$, and $y_3 = 1$:
   a) Compute the Wronskian $W(y_1, y_2, y_3)$.

$$W(y_1,y_2,y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \ (2) = \begin{vmatrix} e^x & e^{-x} & 1 \\ e^x & -e^{-x} & 0 \\ e^x & +e^{-x} & 0 \end{vmatrix} \ (2) = e^x \begin{vmatrix} -e^{-x} & 0 \\ e^{-x} & 0 \end{vmatrix} - e^{-x} \begin{vmatrix} e^x & 0 \\ e^x & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^x \end{vmatrix} \ (2)$$

$$= e^x \cdot 0 - e^{-x} \cdot 0 + 1 \cdot [e^x \cdot e^{-x} - e^x(-e^{-x})] = 1 \cdot [e^0 + e^0] \ (2) = 1+1 = \boxed{2}.$$

   b) Are the three functions linearly independent? Explain your answer.

(2) Yes, because their Wronskian is not zero.

4. *(25 points)*

[5] a) Check that $y_p = 2x - 1$ is a solution of $y'' + y' - 2y = 4 - 4x$.

$y_p' = 2$ (1) and $y_p'' = 0$ (1), so $y_p'' + y_p' - 2y_p = 0 + 2 - 2(2x-1)$ (2)

$= 2 - 4x + 2 = 4 - 4x$. (1)

[8] b) Find the general solution of $y'' + y' - 2y = 0$.

The characteristic equation is $r^2 + r - 2 = 0$ (2), or $(r+2)(r-1) = 0$, (2)
with roots $r = -2$ and $r = 1$ (2), so the general solution is

$$\boxed{y = C_1 e^{-2x} + C_2 e^x} \ (2)$$

[12] c) Find the solution of the equation $y'' + y' - 2y = 4 - 4x$ which satisfies the initial conditions $y(0) = 0$ and $y'(0) = 0$.

The general solution of the inhomogeneous equation is
$y = y_c + y_p$ (2), where y_p can be taken as $2x - 1$ (from part a) and
y_c as $C_1 e^{-2x} + C_2 e^x$ (from part b). So

$$y = C_1 e^{-2x} + C_2 e^x + 2x - 1 \ (2), \text{ and}$$
$$y' = -2C_1 e^{-2x} + C_2 e^x + 2 \ (2).$$

Then $y(0) = 0 \Rightarrow C_1 + C_2 - 1 = 0 \Rightarrow C_1 + C_2 = 1$ (1)
$y'(0) = 0 \Rightarrow -2C_1 + C_2 + 2 = 0 \Rightarrow -2C_1 + C_2 = -2$ (1)
Subtracting gives $3C_1 = 3$, or $C_1 = 1$ and so $C_2 = 0$. (2)

Thus $\boxed{y = e^{-2x} + 2x - 1}$ (2)

5. *(20 points)* Find the solution of the equation $y'' - 6y' + 9y = 0$ which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 8$.

The characteristic equation is ② $r^2 - 6r + 9 = 0$ or ② $(r-3)^2 = 0$, with the real repeated root ② $r = 3$, so the general solution is $y = C_1 e^{3x}$ ② $+ C_2 x e^{3x}$. ②

Then $y' = 3C_1 e^{3x} + C_2[1 \cdot e^{3x} + x \cdot 3e^{3x}]$ ②

or $y' = [3C_1 + C_2]e^{3x} + 3C_2 x e^{3x}$.

$y(0) = 1 \Rightarrow C_1 + 0 = 1$, ~~...~~ or $C_1 = 1$, ② and

$y'(0) = 8 \Rightarrow 3C_1 + C_2 = 8$, so $3 + C_2 = 8$, ② and $C_2 = 5$. ②

Hence $\boxed{y = e^{3x} + 5xe^{3x}}$ ②.

As I mentioned at the exam I meant to put 5y here. ~~...~~ The problem with -5 instead is solved on the next page.

6. *(15 points)* Given that $y = e^x$ is a solution of the equation $y''' - 3y'' + 7y' - 5 = 0$, find the general solution.

The characteristic equation is ② $r^3 - 3r^2 + 7r - 5 = 0$, and since $y = e^{1 \cdot x}$ is a solution of the differential equation, then $r = 1$ is a solution of the characteristic equation. So $r - 1$ is a ② factor of $r^3 - 3r^2 + 7r - 5$. Dividing, we get

$$
\begin{array}{r|l}
 & r^2 - 2r + 5 \\
\hline
r - 1 & r^3 - 3r^2 + 7r - 5 \\
 & \underline{r^3 - r^2} \\
 & -2r^2 + 7r \\
 & \underline{-2r^2 + 2r} \\
 & 5r - 5 \\
 & \underline{5r - 5} \\
 & 0
\end{array}
$$

② , so $(r-1)(r^2 - 2r + 5) = 0$.

The roots of $r^2 - 2r + 5 = 0$ are $r = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 5 \cdot 1}}{2}$ ② $=$

$= \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm \dfrac{\sqrt{16}\, i}{2} = 1 \pm \dfrac{4i}{2} = 1 \pm 2i$ ②, corresponding to solutions

$e^x \cos 2x$ and $e^x \sin 2x$ ②. So the general solution is ②

$$\boxed{y = C_1 e^x + C_2 e^x \cos 2x + C_3 e^x \sin 2x}$$

$9+12+10+1+1+1$

9

For $y''' - 3y'' + 7y' = 5$, The char. equation is $r^3 - 3r^2 + 7r = 0$ 2

or 2 $r(r^2 - 3r + 7) = 0$,

with roots $r = 0$ 1 and $r = \dfrac{3 \pm \sqrt{9-28}}{2}$ 2 $= \dfrac{3}{2} \pm \dfrac{\sqrt{19}\, i}{2}$

So $y_c = C_1 e^{0x} + C_2 e^{\frac{3}{2}x} \cos\left(\frac{\sqrt{19}}{2}x\right) + C_3 e^{\frac{3}{2}x} \sin\left(\frac{\sqrt{19}}{2}x\right)$ 2

Since $C_1 e^{0x} = C_1$ is in y_c, which duplicates the constant 5 on the right-hand side of the inhomogeneous equation, we have to take $y_p = Ax$ 2 and solve for A.

Then $y_p' = A$, $y_p'' = 0$, $y_p''' = 0$, so we get

$0 - 3\cdot 0 + 7\cdot A = 5 \Rightarrow A = \frac{5}{7}. \Rightarrow y_p = \frac{5}{7}x$ 2

So the general solution of the inhomogeneous equation is 2

$$\boxed{y = y_c + y_p = C_1 + C_2 e^{\frac{3}{2}x} \cos\left(\frac{\sqrt{19}}{2}x\right) + C_3 e^{\frac{3}{2}x} \sin\left(\frac{\sqrt{19}}{2}x\right) + \frac{5}{7}x}$$

Since $y = e^x$ is not a solution of $y''' - 3y'' + 7y' = 5$ (as can be seen by substitution: $e^x - 3e^x + 7e^x \neq 5$!), The problem as I stated it on the exam is not correct. No one noticed this till after several people had already left, so if you assumed it was correct and solved the problem accordingly, that's OK.