EXAM 3
Math 3413
11-22-13

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* In this question you are asked to use Laplace transforms to solve the initial-value problem

$$x'' + 9x = \delta(t-2), \quad x(0) = 5, \quad x'(0) = 7.$$

a) Find an equation for the Laplace transform $X(s)$ of $x(t)$.

$$s^2 X(s) - s\cdot x(0) - x'(0) + 9X(s) = \mathcal{L}\{\delta(t-2)\} = e^{-2s} \quad (1)\ (1)\ (2)$$

$$s^2 X(s) - 5s - 7 + 9X(s) = e^{-2s} \quad (1)\ (1)$$

[10]

$$(s^2+9)X(s) = 5s + 7 + e^{-2s} \quad (1)\ (1)$$

b) Solve the equation in part *a)* to find $X(s)$.

(2) $$X(s) = \frac{5s}{s^2+9} + \frac{7}{s^2+9} + \frac{e^{-2s}}{s^2+9}$$

[10] *c)* Use your answer to part *b)* to find $x(t)$.

We have $x(t) = \mathcal{L}^{-1}\left\{\frac{5s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{7}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+9}\right\}$;

and $\mathcal{L}^{-1}\left\{\frac{5s}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = 5\cos(3t)$, (2)

$\mathcal{L}^{-1}\left\{\frac{7}{s^2+9}\right\} = \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{7}{3}\sin(3t)$, (2)

and $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\}$, where $F(s) = \frac{1}{s^2+9}$ (2), $f(t) = \frac{1}{3}\sin(3t)$.

$= u(t-2)f(t-2)$ (2)

$= u(t-2)\cdot\frac{1}{3}\sin(3(t-2))$

(2)

so $$x(t) = \boxed{5\cos(3t) + \frac{7}{3}\sin(3t) + u(t-2)\cdot\frac{1}{3}\sin(3(t-2))}$$

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2. *(15 points)* In this question you are asked to use Laplace transforms to find a formula for the solution of the initial-value problem

$$x''(t) - x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t)$ is a given function. Use $F(s)$ to stand for the Laplace transform of $f(t)$.

a) Find an equation for the Laplace transform $X(s)$ of $x(t)$.

[7]

$$s^2 X(s) - s\,x(0) - x'(0) - X(s) = \mathcal{L}\{f(t)\} = F(s) \quad 2$$

$$s^2 X(s) - 0 - 0 - X(s) = F(s)$$

$$(s^2 - 1)\, X(s) = F(s) \quad 3$$

b) Solve the equation in part *a)* to find $X(s)$.

$$X(s) = \frac{1}{s^2-1} \cdot F(s) \quad 2$$

c) From your answer to part *b)*, find a formula for $x(t)$ using the convolution theorem. Your answer should be an integral involving f.

[8]

Let $G(s) = \frac{1}{s^2-1}$, then $g(t) = \sinh(t)$. 2

By the convolution Theorem,

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1} \cdot F(s)\right\} = \mathcal{L}^{-1}\{F(s) \cdot G(s)\}$$

2

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau \quad 2$$

$$\boxed{= \int_0^t f(\tau) \sinh(t-\tau)\, d\tau} \quad 2$$

3. *(10 points)* Suppose $f(t) = \begin{cases} 0, & 0 < t < 7 \\ (t-7)^3, & t \geq 7 \end{cases}$. Find the Laplace transform of $f(t)$.

2

$$f(t) = u(t-7) \cdot (t-7)^3, \quad 2$$

2

So $f(t) = u(t-7) \cdot g(t-7)$ where $g(t) = t^3$

So $\mathcal{L}\{f(t)\} = e^{-7s} G(s) = e^{-7s} \mathcal{L}\{g(t)\} = e^{-7s}\left(\frac{3!}{s^4}\right) = \boxed{\frac{6e^{-7s}}{s^4}}$.

2 2

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4. *(15 points)* Use the method of power series to solve the differential equation

$$y' - 5xy = 0.$$

Assume a solution of the form

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \dots,$$

where c_0 is arbitrary, and find the values of c_1, c_2, c_3, and c_4 in terms of c_0. (Some of them will be zero, others will be multiples of c_0.)

$c_1 =$ 0

$c_2 =$ $5c_0/2$

$c_3 =$ 0

$c_4 =$ $25c_0/8$

$$y' = 0 + c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots \quad 3$$

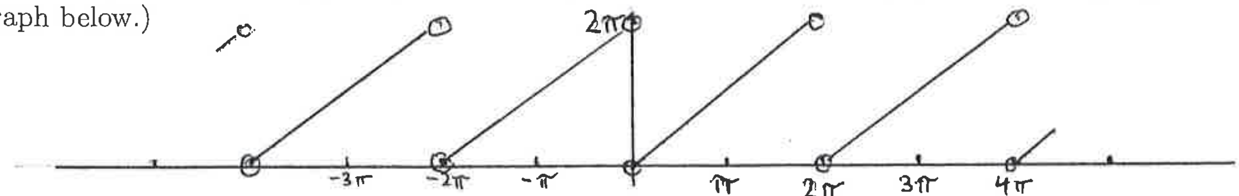
and $$5xy = 5c_0 x + 5c_1 x^2 + 5c_2 x^3 + \dots \quad 2$$

Subtracting,

$$0 = y' - 5xy = c_1 + (2c_2 - 5c_0)x + (3c_3 - 5c_1)x^2 + (4c_4 - 5c_2)x^3 + \dots, \quad 2$$

so
$$\begin{cases} 0 = c_1 \\ 0 = 2c_2 - 5c_0 \\ 0 = 3c_3 - 5c_1 \\ 0 = 4c_4 - 5c_2 \end{cases},$$
hence
$$\begin{cases} 0 = c_1, & 2 \\ c_2 = \frac{5c_0}{2}, & 2 \\ c_3 = \frac{5c_1}{3} = \frac{5}{3}\cdot 0 = 0, & 1 \quad 1 \\ c_4 = \frac{5c_2}{4} = \frac{5}{4}\left(\frac{5c_0}{2}\right) = \frac{25}{8}c_0. & 1 \quad 1 \end{cases}$$

5. *(20 points)* The function $f(t)$ is periodic with period 2π, and is given by $f(t) = t$ for $0 \le t \le 2\pi$. (See its graph below.)



a) Find the Fourier series for f. You may use the formulas

[16]

$$\int t\sin at\ dt = -\frac{1}{a}t\cos at + \frac{1}{a^2}\sin at, \qquad \int t\cos at\ dt = \frac{1}{a}t\sin at + \frac{1}{a^2}\cos at.$$

Write out the first four terms of the series. Take $L = \pi$. (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{2\pi} t\,dt = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{1}{\pi}\cdot\frac{4\pi^2}{2} = 2\pi. \quad (1)\ (1)\ (1)$$

For $n>0$, $a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos(nt)\,dt =$ (1)

$$= \frac{1}{\pi}\left[\frac{1}{n}t\sin(nt) + \frac{1}{n^2}\cos(nt)\right]_0^{2\pi} \quad (1)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}2\pi\cancel{\sin}(2\pi n) + \frac{1}{n^2}\cos(2\pi n) - \left(\frac{1}{n}\cdot 0 + \frac{1}{n^2}\cos(0)\right)\right] \quad (1)$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\cos(2\pi n) - \frac{1}{n^2}\cos(0)\right] = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0.$$

For $n>0$,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin(nt)\,dt = \frac{1}{\pi}\left[-\frac{1}{n}t\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_0^{2\pi} \quad (1)\ (1)$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}2\pi\cos(2\pi n) + \frac{1}{n^2}\cancel{\sin}(2\pi n) - \left(-\frac{1}{n}\cdot 0 + \frac{1}{n^2}\cancel{\sin}(0)\right)\right] \quad (1)\ (1)$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n}\right] = \frac{-2}{n}.$$

So $f(t) \sim \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin(nt) = \boxed{\pi - 2\sin t - \frac{2}{2}\sin(2t) - \frac{2}{3}\sin(3t) - \cdots}$ (1) (1) (1) (1)

b) What value does the Fourier series converge to for $t = \pi/2$? Briefly explain your answer.

[4] Since f is piecewise smooth and is continuous at $t = \frac{\pi}{2}$,

then the Fourier series converges to $f(t)$ at $t = \frac{\pi}{2}$.

But $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, so the Fourier series converges to $\frac{\pi}{2}$.

(Your answer does not have to be quite this complete)

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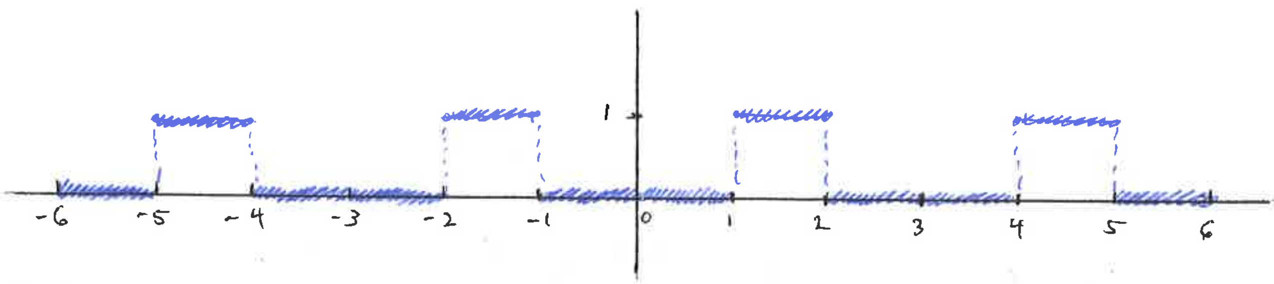
6. *(20 points)* A function $f(t)$ is even, is periodic with period 6, and is given for $0 < t < 3$ by

$$f(t) = \begin{cases} 0, & 0 < t < 1; \\ 1, & 1 < t < 2; \\ 0, & 2 < t < 3. \end{cases}$$

a) Sketch the graph of $f(t)$ for $-6 < t < 6$.

[4]



b) Find the Fourier series of f. If there are trig functions in the formulas for the Fourier coefficients, you do not need to evaluate them.

[12]

Since f is even, we know that $b_n = 0$ for all $n \geq 1$. 3

For $n=0$, we have, taking $L=3$ 1, and using that f is even,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{3}\int_0^3 f(t)\,dt = \frac{2}{3}\int_1^2 1\,dt$$
1 1

$$= \frac{2}{3}\Big[t\Big]_1^2 = \frac{2}{3}.$$ 1

Similarly, for $n \geq 1$ we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{3}\right)dt = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{3}\right)dt =$$

2 1

$$= \frac{2}{3}\int_1^2 1\cos\left(\frac{n\pi t}{3}\right)dt = \frac{2}{3}\left[\frac{\sin\left(\frac{n\pi t}{3}\right)}{(n\pi/3)}\right]_1^2$$

1

$$= \frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{n\pi}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right] = \frac{2}{n\pi}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right]$$

1

So $$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi t}{3}\right) = \boxed{\frac{1}{3} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right]\cos\left(\frac{n\pi t}{3}\right)}$$

c) What value does the Fourier series converge to for $t = 1$? Briefly explain your answer.

[4]

Since f has a jump discontinuity at $t=1$, with $f(1^-)=0$ and $f(1^+)=1$,

Then the Fourier series converges at $t=1$ to $\frac{1}{2}\left[f(1^-)+f(1^+)\right] = \frac{0+1}{2} = \boxed{\frac{1}{2}}$.