Math 4073 — Numerical Analysis
Assignment 2

1.

a) The golden section search proceeds by a procedure similar to the bisection method, and therefore has an error estimate similar to that for the bisection method. What is this error estimate for the golden section method? Explain your answer.

b) The function $f(x) = x^4 - 3x^2 + x$ has two relative minima, one of which is an absolute minimum. Choose an appropriate starting interval $[a, b]$ to find each of these minima, and use your error estimate from part (a) to estimate how many iterations it will take to approximate the solution to within 10^{-6} using your choice of a and b. Use the MATLAB program golden_section_search.m to carry out this many steps and obtain the approximate solution for each minimum. (You will want to modify the script myfunction.m accordingly.) Include a printout of the successive approximations computed.

2. This problem justifies the modified Newton method for roots of multiplicity 2.

(a) Suppose $f(x)$ has a root of multiplicity 2 at $x = p$, so $f(p) = 0$ and $f'(p) = 0$, but $f''(p) \neq 0$. Define $g(x)$ by

$$g(x) = x - \frac{2f(x)}{f'(x)} \quad \text{whenever } x \neq p,$$

and

$$g(p) = p.$$

Show that $g'(p) = 0$ by showing that

$$\lim_{x \to p} \frac{g(x) - g(p)}{x - p} = 0.$$

Hint: use L'hopital's rule (you will have to use it twice).

(b) Explain why it follows from (a) that if f has a root of multiplicity 2 at p and we define

$$p_{n+1} = p_n - \frac{2f(p_n)}{f'(p_n)} \quad \text{for } n = 1, 2, 3, \ldots$$

then p_n will converge quadratically to p, if p_0 is close enough to p. (Cite a theorem from class.)

3. (This problem justifies Aitken's Δ^2 method.) Suppose p_n is a sequence which converges linearly to p, so

$$\lim_{n \to \infty} \frac{p_{n+1} - p}{p_n - p} = \lambda$$

for some constant λ. Define

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{(p_{n+2} - p_{n+1}) - (p_{n+1} - p_n)},$$

and prove that

$$\lim_{n \to \infty} \frac{\hat{p}_n - p}{p_n - p} = 0.$$

Note: I recommend trying to do this problem by yourself first without looking at the hint below, as a good way to gain experience in using algebra wisely. The trick is to find the right way to express $\frac{\hat{p}_n - p}{p_n - p}$ in order to easily compute the limit. Once you find the right way, the problem becomes pretty easy.

(Hint: first show that

$$\frac{\hat{p}_n - p}{p_n - p} = 1 - \frac{\left[\frac{p_{n+1}-p}{p_n-p} - 1\right]^2 \frac{p_n-p}{p_{n+1}-p}}{\left[\frac{p_{n+2}-p}{p_{n+1}-p} - 1\right] - \left[1 - \frac{p_n-p}{p_{n+1}-p}\right]},$$

then take the limit of both sides as $n \to \infty$.)

4. Find the Lagrange form for the interpolating polynomial for the function $f(x) = \sin(\ln x)$ with the three data points $x_0 = 2$, $x_1 = 2.4$, and $x_2 = 2.6$; so here $n = 2$. Use Theorem 3.3 from the text to estimate the maximum error on the interval $[2, 2.6]$. We will go over how to do this in class, but Example 3 in section 3.1 provides an example if you want to do this problem before then. You do not need to get a precise estimate on the error; it is enough to show that the maximum error is less than 0.025.

(Note: it is interesting to graph the difference between $f(x)$ and the interpolating polynomial on the interval $[2, 2.6]$ to see how small it really is. You don't have to turn that graph in, though.)

5.

(a) Use the method of divided differences to find Newton's form of the interpolating polynomial of degree three which interpolates the data

x :	3	7	1	2
$f(x)$:	10	146	2	1 .

(By this I mean that $x_0 = 3$, $x_1 = 7$, $x_2 = 1$, and $x_3 = 1$; and $f(x_0) = 10$, $f(x_1) = 146$, $f(x_2) = 2$, and $f(x_3) = 1$.)

(b) Same as part (a), but now use the data

x :	3	7	1	2
$f(x)$:	12	146	2	1 .