Problem 9, page 23

Here is one way to do the second part of the problem (from Huy Le). First, put $z = e^{i\theta}$ into the formula from the first part of the problem,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

to get

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Now rewrite the right side of the equation using Euler's identity, and then multiply by the conjugate of the denominator, to get

$$\begin{aligned} 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{in\theta} &= \frac{1 - \cos(n+1)\theta - i\sin(n+1)\theta}{1 - \cos\theta - i\sin\theta} \\ &= \frac{1 - \cos(n+1)\theta - i\sin(n+1)\theta}{1 - \cos\theta - i\sin\theta} \cdot \frac{1 + \cos\theta + i\sin\theta}{1 + \cos\theta + i\sin\theta}. \end{aligned}$$

Taking the real part of both sides, we get

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{[1 - \cos(n+1)\theta](1 - \cos\theta) + \sin(n+1)\theta\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta}.$$

Now we rewrite the last expression using various trigonometric identities. We get

$$\begin{aligned} &\frac{[1 - \cos(n+1)\theta](1 - \cos\theta) + \sin(n+1)\theta\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta} \\ &= \frac{1 - \cos\theta - \cos(n+1)\theta + \cos n\theta}{2 - 2\cos\theta} = \frac{1}{2} - \frac{\cos(n+1)\theta - \cos n\theta}{4\sin^2\frac{\theta}{2}} \\ &= \frac{1}{2} - \frac{-2\sin\frac{2n+1}{2}\theta\sin\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}. \end{aligned}$$

Putting this together with the previous equation gives

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}},$$

as desired.

(Can you tell which trigonometric identities were used here? They include the identities $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$ and $\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$.)