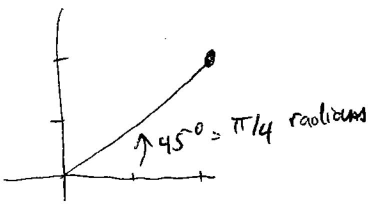
EXAM 3 **Math 4103** **4-29-11** Name key

1. *(20 points)* Evaluate (find the real and imaginary part):

[10] **a.** $\mathrm{Log}(2+2i)$

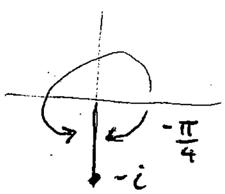


$\mathrm{Log}(2+2i) = \ln|2+2i| + i\,\mathrm{Arg}(2+2i)$ 4

$= \ln\sqrt{2^2+2^2} + i\,\pi/4$ 3 3

$= \boxed{\ln\sqrt{8} + \frac{i\pi}{4}}$

[10] **b.** $(-i)^{1+2i}$ ← Find principal value



$(-i)^{1+2i} = e^{(1+2i)\,\mathrm{Log}(-i)}$. 3

$\mathrm{Log}(-i) = \ln|-i| + i\,\mathrm{Arg}(-i) = \ln 1 + i\left(-\frac{\pi}{2}\right) = -\frac{\pi i}{2}$. 3

So the principal value of $(-1)^{1+2i}$ is

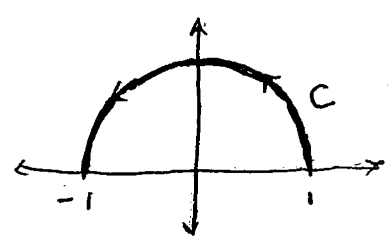
$e^{(1+2i)\left(-\frac{\pi i}{2}\right)} = e^{-\frac{\pi i}{2}}\, e^{\pi}$ 2

$= e^{\pi}\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$

$= e^{\pi}(-i) = \boxed{-i\,e^{\pi}}$ 2

2. *(20 points)*

[10] **a.** Evaluate $\int_C |z|^2\, dz$, where C is the upper half of the circle of radius 1 centered at the origin, traversed counterclockwise (see diagram).



We can parametrize C as

$z(t) = e^{it}$ 2, $0 \le t \le \pi$. 2

Then $|z|^2 = |e^{it}|^2 = 1$, 2

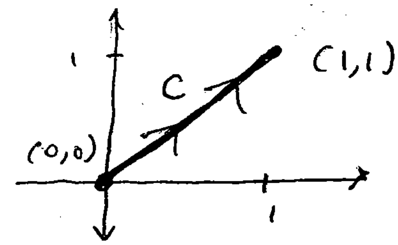
So $\int_C |z|^2\, dz = \int_0^{\pi} 1\; z'(t)\, dt$ 2

$= \int_0^{\pi} i e^{it}\, dt = \left[e^{it}\right]_{t=0}^{t=\pi} = e^{i\pi} - e^{0}$

$= (-1) - 1 = \boxed{-2}$

2

[10] **b.** Evaluate $\int_C \bar{z}\, dz$, where C is the line segment from $(0,0)$ to $(1,1)$ (see diagram).



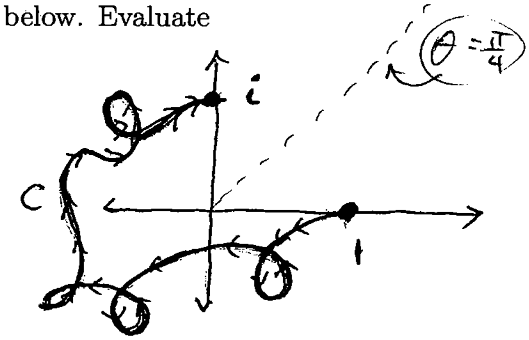
C is part of the line $y=x$, so we can parametrize C as: $\begin{cases} y=t \\ x=t \end{cases}$ $0\le t\le 1$. (2)

Then $\int_C \bar{z}\,dz = \int_0^1 (x-iy)\left(\frac{dx}{dt}+i\frac{dy}{dt}\right)dt$

$= \int_0^1 (t-it)(1+i\cdot 1)\,dt = \int_0^1 t(1-i)(1+i)\,dt$

$= (1-i)(1+i)\int_0^1 t\,dt = 2\int_0^1 t\,dt = \left[t^2\right]_0^1 = 1.$

3. *(20 points)* Suppose C is the contour shown in the diagram below. Evaluate



[10] **a.** $\int_C \frac{1}{z^2}\,dz$

Here $f(z)=z^{-2}$ has the antiderivative $F(z) = -z^{-1} = \frac{-1}{z}$, defined and analytic on the domain $\mathbb{C}\setminus\{0\}$, which includes C. So

$$\int_C \frac{1}{z^2}\,dz = F(i)-F(1) = \frac{-1}{i} - \left(\frac{-1}{1}\right) = \boxed{i+1.}$$

[10] **b.** $\int_C \frac{1}{z}\,dz$

To find an antiderivative of $f(z)=\frac{1}{z}$ on a domain which includes C, we can use the branch of the logarithm function defined by:

$\log z = \ln|r| + i\theta$ where $z=re^{i\theta}$ and $\frac{\pi}{4} < \theta < \frac{\pi}{4}+2\pi$.

(This branch is defined and analytic on the domain obtained by removing the origin and the dotted line in the diagram above).

Then $\int_C \frac{1}{z}\,dz = \log(i) - \log(1)$, and we evaluate $\log(i)$ as $\ln 1 + i\left(\frac{\pi}{2}\right) = i\frac{\pi}{2}$ and $\log(1)$ as $\ln 1 + i(2\pi)$. So

$$\int_C \frac{1}{z}\,dz = \frac{i\pi}{2} - i(2\pi) = \boxed{-\frac{3\pi i}{2}}$$

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4. *(25 points)* Let C be the circle of radius 5 centered at the origin, with positive orientation. Evaluate the integrals, and state which theorem or formula you are using.

[7] **a.** $\int_C \frac{z^3}{z-10}\,dz$ The point where $\frac{z^3}{z-10}$ fails to be analytic is $z=10$, 5 which is outside C. So $\frac{z^3}{z-10}$ is analytic on and within C, so by Cauchy's Theorem the integral equals 0. 2

[8] **b.** $\int_C \frac{z^3}{z-3}\,dz$ The point $z=3$ is within C, 2 so we can use 2 the Cauchy formula $2\pi i\, f(3) = \int_C \frac{f(z)}{z-3}\,dz$ 2 with $f(z)=z^3$.

Thus $\int_C \frac{z^3}{z-3}\,dz = 2\pi i \cdot 3^3 = \boxed{54\pi i}$ 2

[9] **c.** $\int_C \frac{z^3}{(z-3)^3}\,dz$ We use the extension of the Cauchy integral formula: $\int_C \frac{f(z)}{(z-z_0)^{n+1}} = 2\pi i\,\frac{f^{(n)}(z_0)}{n!}$ 2 2, with $z_0=3$ and $n=2$. 1 Thus 1

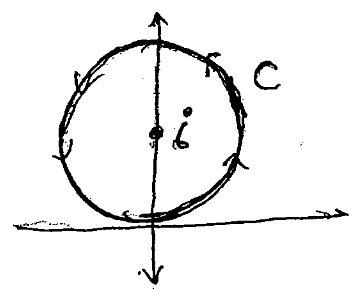
$\int_C \frac{z^3}{(z-3)^3}\,dz = \frac{2\pi i\, f''(3)}{2!} = \pi i\, f''(3)$. 1 Here $f(z)=z^3$, $f'(z)=3z^2$, $f''(z)=6z$, so $f''(3)=18$ 1 and the answer is $\boxed{18\pi i}$. 1

5. *(15 points)* Evaluate $\int_C \frac{1}{z^2+1}\,dz$ if

a. C is the positively oriented circle of radius 1 centered at i.

Use ~~[illegible]~~ $\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$ 3 with

$f(z) = \frac{1}{z+i}$ and $z_0 = i$ 2 (Thus $\frac{f(z)}{z-z_0} = \frac{\left(\frac{1}{z+i}\right)}{z-i}$.)

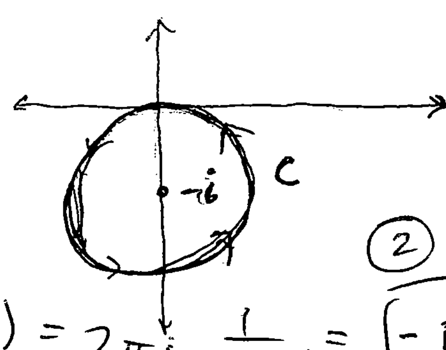


So $\int_C \frac{1}{z^2+1}\,dz = \int_C \frac{1}{(z+i)(z-i)}\,dz$ 2 $= \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) = 2\pi i\left(\frac{1}{i+i}\right) = \boxed{\pi}$ 2

b. C is the positively oriented circle of radius 1 centered at $-i$.

Here we must take $z_0 = -i$ and $f(z) = \frac{1}{z-i}$, 2

So $\int_C \frac{1}{z^2+1}\,dz = \int_C \frac{1}{(z+i)(z-i)}\,dz =$



$= \int_C \frac{f(z)}{z+i}\,dz = \int_C \frac{f(z)}{z-(-i)}\,dz$ 2 $= 2\pi i\, f(-i) = 2\pi i\,\frac{1}{(-i-i)} = \boxed{-\pi}$. 2