FINAL EXAM
Math 4103
5-12-11

Name key

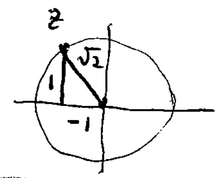
1. *(20 points)*

[10] **a.** Find the real and imaginary parts of $\dfrac{2+i}{2-i}$. Show all work.

$$\frac{2+i}{2-i} = \frac{(2+i)(2+i)}{(2-i)(2+i)} = \frac{4-1+4i}{4+1} = \frac{3}{5} + \frac{4i}{5}$$

3 3 3 1

[10] **b.** If $z = \dfrac{-1+i}{\sqrt{2}}$, find the real and imaginary parts of $\dfrac{1}{z^{18}}$. Show all work.



$z = e^{3\pi i/4}$ (2) (see diagram), so

$$\frac{1}{z^{18}} = z^{-18} = e^{(3\pi i/4)(-18)} = e^{-27\pi i/2} = e^{-13\pi i} e^{-\pi i/2} = (-1)e^{-\pi i/2}$$

2 2

$= (-1)(-i) = i$

(So $\mathrm{Re}(1/z^{18}) = 0$ (2) and $\mathrm{Im}(1/z^{18}) = 1$.) (2)

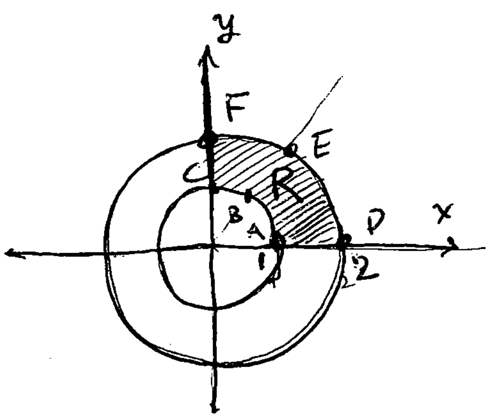
2. *(20 points)* The points A, B, C, D, E, F are given by

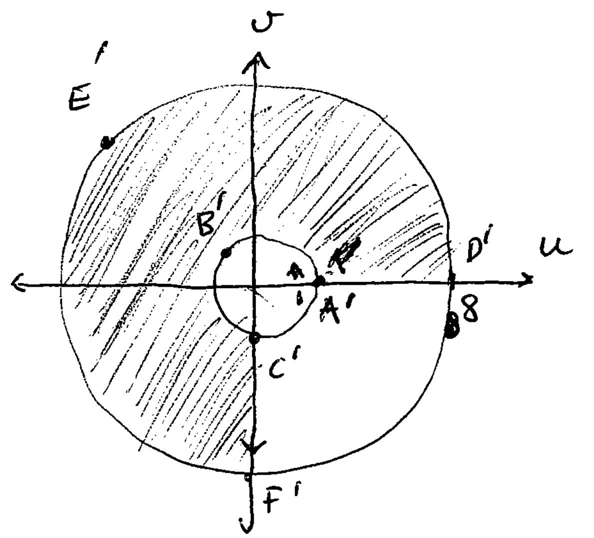
$$A = 1, B = e^{i\pi/4}, C = e^{i\pi/2}$$
$$D = 2, E = 2e^{i\pi/4}, F = 2e^{i\pi/2}$$

[6] **a.** Mark the points A, B, C, D, E, F in the z-plane (in the diagram below). at left

[10] **b.** Mark their images A', B', C', D', E', F' in the w-plane (in the diagram below at right) if $w = z^3$.

[4] **c.** Find the image of the shaded region R in the w-plane, and shade it in.





3. *(20 points)*

[10] **a.** Find all three values of $(8i)^{1/3}$ (in exponential form) $2e^{\pi i/6},\ 2e^{\frac{5\pi i}{6}},\ 2e^{\frac{3\pi i}{2}}$

[4] **b.** Of the three values of in part **a**, which is the principal root? $2e^{\pi i/6}$ (4)

[6] **c.** Graph the three roots in the complex plane.

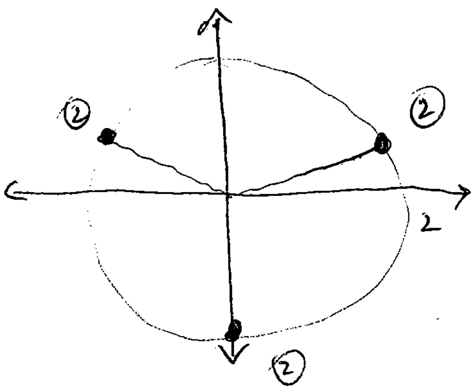
(5)

$$8i = 8e^{\pi i/2} = 8e^{\pi i/2 + 2\pi i} = 8e^{\frac{\pi i}{2}+4\pi i}$$

$$(8i)^{\frac{1}{3}} = \left\{2e^{\pi i/6},\ 2e^{\pi i/6+\frac{2\pi i}{3}},\ 2e^{\frac{\pi i}{6}+\frac{4\pi i}{3}}\right\}$$

(5)

$$= \left\{2e^{\pi i/6},\ 2e^{\frac{5\pi i}{6}},\ 2e^{\frac{3\pi i}{2}}\right\}$$



4. *(12 points)*

[5] **a.** Find all the possible values of $\log(e^{9\pi i/4})$.

$e^{9\pi i/4} = e^{9\pi i/4 + 2\pi i n}$ $(n \in \mathbb{N})$, so $\log\left(e^{9\pi i/4}\right) = \left\{\frac{9\pi i}{4} + 2\pi i n : n \in \mathbb{N}\right\}$

[3] **b.** Find $\text{Log}(e^{9\pi i/4})$.

We have $-\pi \le \text{Log}\, e^{\frac{9\pi i}{4}} < \pi$, so $\text{Log}\, e^{\frac{9\pi i}{4}} = \frac{9\pi i}{4} - 2\pi i = \boxed{\frac{\pi i}{4}}$

c. Find all the possible values of $e^{\log(9\pi i/4)}$.

[4] The only possible value is $\frac{9\pi i}{4}$.

5. *(20 points)* Suppose $f(z)$ is defined for all $z \in \mathbf{C}$ by $f(x+iy) = y + ix$. Show that f is not differentiable at $z = 0$ by showing that the limit

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

does not exist.

Letting $\Delta z = \Delta x + i\Delta y$ (5) with $\Delta y = 0$, we get $\frac{f(0+\Delta z) - f(0)}{\Delta z} = \frac{f(\Delta x + i0) - f(0)}{\Delta x} =$

$= \frac{0 + i\Delta x - 0}{\Delta x} = i$ (5), so as $\Delta z \to 0$ we get $\frac{f(0+\Delta z) - f(0)}{\Delta z} \to \boxed{i}$

Letting $\Delta z = \Delta x + i\Delta y$ (5) with $\Delta x = 0$, we get $\frac{f(0+\Delta z) - f(0)}{\Delta z} = \frac{f(0+i\Delta y) - f(0)}{i\Delta y} =$

$= \frac{\Delta y + i0 - 0}{i\Delta y} = \frac{1}{i} = -i$ (5), so as $\Delta z \to 0$ we get $\frac{f(0+\Delta z) - f(0)}{\Delta z} \to \boxed{-i}$.

Since $(f(0+\Delta z) - f(0))/\Delta z$ does not have a unique limit as $\Delta z \to 0$, $\boxed{f \text{ is not diff. at } 0}$.

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6. *(20 points)* Suppose $f(z)$ is defined for all $z \in \mathbf{C}$ by $f(x+iy) = e^{2x}(\cos^2 y - \sin^2 y) + i(2e^{2x}\sin y \cos y)$. Show that f is entire (analytic on $\mathbf{C}$).

(2) (2)

$f = u + iv$ with $u = e^{2x}(\cos^2 y - \sin^2 y)$ and $v = 2e^{2x}\sin y \cos y$.

(3) (3)

Then $u_x = 2e^{2x}(\cos^2 y - \sin^2 y) = v_y \left(= 2e^{2x}(\cos y \cdot \cos y + \sin y(-\sin y))\right)$

(3)

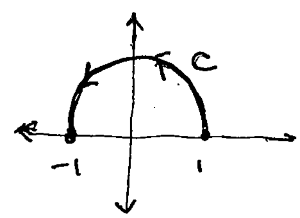
and $u_y = e^{2x}(2\cos y(-\sin y) - 2\sin y \cos y) = e^{2x}(-4\sin y \cos y)$

(3)

$= -(4e^{2x}\sin y \cos y) = -v_x,$

so u and v satisfy the Cauchy-Riemann equations for all (4) x and y. Since u and v are clearly C^1, it follows that f is analytic on all of $\mathbb{C}$.

7. *(10 points)* Evaluate $\int_C \bar{z}\, dz$, where C is the upper half of the circle of radius 1 centered at the origin, traversed counterclockwise (see diagram).



Take $z = e^{it}$ for $0 \le t \le \pi$. (2)

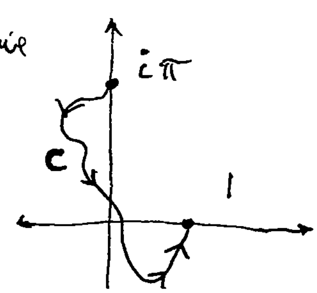
Then $\bar{z} = e^{-it}$ (2) and $\frac{dz}{dt} = ie^{it}\, dt$, (2)

(2)

so $\int_C \bar{z}\, dz = \int_0^{\pi} e^{-it}(ie^{it})\, dt = \int_0^{\pi} i\, dt$

$= [it]_{t=0}^{t=\pi} = \boxed{\pi i.}$ (2)

8. *(10 points)* Evaluate $\int_C e^{3z}\, dz$, where C is the contour shown in the diagram below.



Let $F(z) = \frac{e^{3z}}{3}$ (4), then F is an antiderivative of e^{3z} on $\mathbb{C}$, so

$\int_C e^{3z}\, dz = F(1) - F(i\pi)$ (4)

$= \frac{e^{3\cdot 1}}{3} - \frac{e^{3\pi i}}{3} = \frac{e^3}{3} - \frac{(-1)}{3}$

$= \left[\frac{e^3+1}{3}\right].$ (2)

9. *(20 points)* Let C be the circle of radius 1 centered at the origin, with positive orientation. Evaluate the integrals, and state which theorem or formula you are using.

[12] a. $\int_C \frac{\sin z}{z^4}\, dz$

By the (extended) Cauchy integral formula, if $f(z)$ is analytic on and within C, then $\int \frac{f(z)}{(z-0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(0)$. (4)

Here we can take $n=3$ and $f(z)=\sin z$.

Then $f'(z)=\cos z$, $f''(z)=-\sin z$, $f'''(z)=-\cos z$, so $\int \frac{f(z)}{z^4}\, dz = \frac{2\pi i}{3!} f'''(0)$ (4)

$= \frac{\pi i}{3}(-\cos 0) = \boxed{-\frac{\pi i}{3}}$. (4)

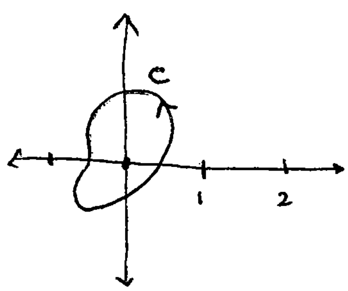
[8] b. $\int_C z^4 \sin z\, dz$

Since $z^4 \sin z$ is entire, then by Cauchy's Theorem (4) it has integral zero over every (simple) closed contour in $\mathbb{C}$.

So $\int_C z^4 \sin z\, dz = 0$. (4)

10. *(20 points)* Evaluate the integral $\int_C \frac{z+1}{z(z-2)}\, dz$ for each of the three positively oriented simple closed contours shown below. Give complete reasons for your answers, stating which theorems or formulas you are using.

[7] a.

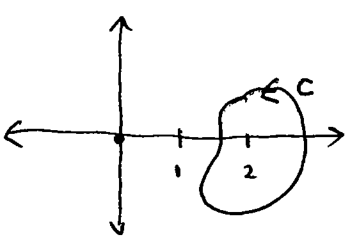


Take $f(z) = \frac{z+1}{z-2}$. (3) Then by the Cauchy integral formula (since f is analytic on and within C),

$\int_C \frac{z+1}{z(z-2)}\, dz = \int_C \frac{f(z)}{z-0}\, dz = 2\pi i f(0)$ (2) (2)

$= 2\pi i\left(\frac{1}{-2}\right) = \boxed{-\pi i}$.

[7] b.

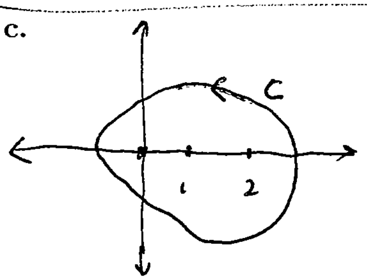


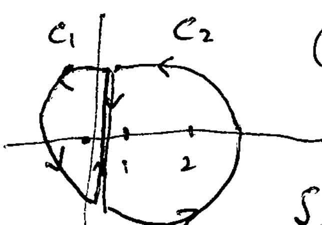
Take $f(z) = \frac{z+1}{z}$. (3) Then f is analytic on and within C, so (2)

$\int_C \frac{f(z)}{z-2}\, dz = 2\pi i\, f(2) = 2\pi i\left(\frac{3}{2}\right) =$ (2)

$= \boxed{3\pi i}$.

[6] c.





(3) Let C_1, C_2 be the contours shown (following the same path as C except cancelling each other along a vertical line between $z=0$ and $z=2$). Then

$\int_C \frac{z+1}{z(z-2)}\, dz = \int_{C_1} \frac{z+1}{z(z-2)}\, dz + \int_{C_2} \frac{z+1}{z(z-2)}\, dz$

$=$ (as above) $(-\pi i) + (3\pi i) = \boxed{2\pi i}$. (3)

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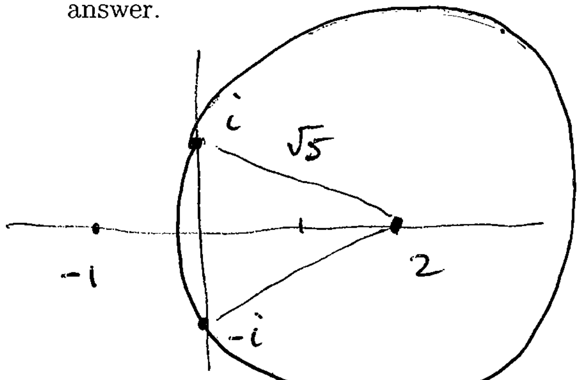
11. *(10 points)* If the function $f(z) = \dfrac{e^z}{(z^2+1)(z+1)}$ were expanded in a Taylor series about the point $z_0 = 2$, what would be the radius of convergence and circle of convergence of the Taylor series? Explain your answer.



By a Theorem from class, The radius of convergence is the distance from $z_0 = 2$ to the nearest singularity of $f(z)$, 3 which in this case is $\sqrt{5}$ 2 (the distance from $z_0=2$ to i or to $-i$) because f has singularities only at -1, i, and $-i$. 3
So the circle of convergence is $\{|z-2|<\sqrt{5}\}$. 2

12. *(18 points)* Expand the following functions in series of the form $\sum a_n z^n$, writing out at least the first three terms of the expansion. Your expansion may include negative powers of z if necessary.

[10] a. $f(z) = \dfrac{1}{1+4z^2}$

We know $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+z^3+\cdots$ for $|z|<1$, 3

and replacing z by $-4z^2$ gives 4

$$\frac{1}{1+4z^2} = \frac{1}{1-(-4z^2)} = \boxed{\sum_{n=0}^{\infty}(-4z^2)^n = 1-4z^2+16z^4-64z^6+\cdots}$$ 3

(for $|4z^2|<1$, or for $|z|<2$).

[8] b. $f(z) = \dfrac{e^z-1}{z^3}$ 4

We know $e^z = \sum_{n=0}^{\infty}\dfrac{z^n}{n!}$ for all $z\in\mathbb{C}$, so

$$\frac{e^z-1}{z^3} = \frac{1}{z^3}\left(\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)-1\right) = \frac{1}{z^3}\left(\sum_{n=1}^{\infty}\frac{z^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{z^{n-3}}{n!}$$ 2 2

$$= \boxed{\frac{1}{z^2}+\frac{1}{2z}+\frac{1}{6}+\frac{z}{24}+\frac{z^2}{120}+\cdots}$$

for all $z\in\mathbb{C}$.