Answer to problem 2.5.1(a)

As we found in class, the general solution to the PDE in 2.5.1(a) with boundary conditions $\frac{\partial u}{\partial x}(0,y)=0$, $\frac{\partial u}{\partial x}(L,y)=0$, and $u(x,0)=0$ is given by

$$u(x,y)=C_0 y+\sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right). \tag{1}$$

To satisfy the boundary condition $u(x,H)=f(x)$, we have to choose the constants C_0 and C_n $(n=1,2,3,\dots)$ correctly.

Putting $y=H$ in equation (1) above, and setting $u(x,H)$ equal to $f(x)$ we get that

$$f(x)=C_0 H+\sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right).$$

Multiplying through by the eigenfunction 1 for the eigenvalue $\lambda=0$ and integrating with respect to x from $x=0$ to $x=L$ gives

$$\int_0^L f(x)\,dx=C_0 H\int_0^L 1\,dx+\sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi H}{L}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx.$$

From the fact that different eigenfunctions are orthogonal, we know that all the integrals on the right-hand side of the equation are zero, except for $\int_0^L 1\,dx$, which is easily seen to be equal to L. So

$$\int_0^L f(x)\,dx=C_0 HL,$$

and solving for C_0 gives $\boxed{C_0=\frac{1}{HL}\int_0^L f(x)\,dx.}$

Similarly, for $m=1,2,3,\dots$, multiplying through equation (1) above by the eigenfunction $\cos\left(\frac{m\pi x}{L}\right)$ and integrating, we get

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)\,dx=C_0 H\int_0^L \cos\left(\frac{m\pi x}{L}\right)\,dx+$$
$$+\sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi H}{L}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx,$$

and again orthogonality gives that all the integrals on the right-hand side are zero, except for the one where $n = m$, and we get

$$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right)\, dx = C_m \sinh\left(\frac{m\pi H}{L}\right) \int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx.$$

Since $\int_0^L \cos^2\left(\frac{m\pi x}{L}\right)\, dx = L/2$, this gives

$$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right)\, dx = C_m \sinh\left(\frac{m\pi H}{L}\right) \frac{L}{2},$$

and solving for C_m gives

$$\boxed{C_m = \frac{2}{L \sinh\left(\frac{m\pi H}{L}\right)} \int_0^L f(s) \cos\left(\frac{m\pi s}{L}\right)\, ds, \text{ for } m = 1, 2, 3, \ldots.}$$