Math 4163
Assignment 10

1. Find the Green's function $G(\mathbf{x};\mathbf{y})$ for the Laplacian in the quarter-plane

$$\Omega = \{(y_1, y_2) : y_1 > 0 \quad \text{and} \quad y_2 > 0\}$$

with Dirichlet boundary conditions. That is, for fixed $\mathbf{x} = (x_1, x_2) \in \Omega$, find the solution of the problem

$$\frac{\partial^2 G}{\partial y_1^2} + \frac{\partial^2 G}{\partial y_2^2} = \delta(\mathbf{x} - \mathbf{y})$$

for $\mathbf{y} = (y_1, y_2) \in \Omega$, with boundary condition

$$G(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for } \mathbf{y} \in \partial\Omega.$$

Here $\partial\Omega$, the boundary of Ω, is the union of the two half lines $\{y_1 = 0 \text{ and } y_2 > 0\}$ and $\{y_2 = 0 \text{ and } y_1 > 0\}$.

This problem is easily done using the method of images. First, for fixed $\mathbf{x} \in \Omega$, write down the Green's function for the Laplacian on the half-plane (call it $H(\mathbf{x}, \mathbf{y})$), which you can find in the lecture notes from Week 14. Then you need to find the right function v to add to $H(\mathbf{x}, \mathbf{y})$ in order to get the desired Green's function $G(\mathbf{x}, \mathbf{y})$. To do this, you should take $v = -H(\tilde{\mathbf{x}}, \mathbf{y})$, where $\tilde{\mathbf{x}}$ is a suitably chosen point outside of Ω.

Describe the correct choice for $\tilde{x}$, write down a formula for $G(\mathbf{x}, \mathbf{y})$ as a function of x_1, x_2, y_1, y_2, and show by direct computation that $G(\mathbf{x}, \mathbf{y}) = 0$ when $\mathbf{y} \in \partial\Omega$ (that is, when either $y_1 = 0$ or $y_2 = 0$).

2. Find the Green's function for the Laplacian in the half-plane

$$\Omega = \{(y_1, y_2) : \quad y_2 > 0\}$$

with Neumann boundary conditions. That is, for fixed $\mathbf{x} \in \Omega$, find the solution of the problem

$$\frac{\partial^2 G}{\partial y_1^2} + \frac{\partial^2 G}{\partial y_2^2} = \delta(\mathbf{x} - \mathbf{y})$$

for $\mathbf{y} = (y_1, y_2) \in \Omega$, with boundary condition

$$\frac{\partial G}{\partial y_2}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for } \mathbf{y} \in \partial\Omega.$$

Here $\partial\Omega$, the boundary of Ω, is the line $\{y_2 = 0\}$.

Again, this is easily done by the method of images. First, for fixed $\mathbf{x} \in \Omega$, write down the Green's function for the Laplacian on the entire plane (call it $\Phi(\mathbf{x}, \mathbf{y})$), which you can find in the lecture notes from Week 14. Then you need to find the right function v to add to $\Phi(\mathbf{x}, \mathbf{y})$ in order to get the desired Green's function $G(\mathbf{x}, \mathbf{y})$. To do this, you should take $v = +\Phi(\tilde{\mathbf{x}}, \mathbf{y})$, where $\tilde{\mathbf{x}}$ is a suitably chosen point outside of Ω.

Describe the correct choice for $\tilde{x}$, write down a formula for $G(\mathbf{x}, \mathbf{y})$ as a function of x_1, x_2, y_1, y_2, and show by direct computation that $\dfrac{\partial G}{\partial y_2}(\mathbf{x}, \mathbf{y}) = 0$ when $\mathbf{y} \in \partial\Omega$ (that is, when $y_2 = 0$).

3. Let Ω be the unit disc in the plane, given in polar coordinates by

$$\Omega = \{(r, \theta) : 0 \le r < 1 \text{ and } -\pi \le \theta \le \pi\}.$$

Earlier in the semester we used separation of variables to solve the boundary-value problem for the Laplacian on Ω:

$$\begin{aligned} \Delta u &= 0 \quad \text{for } (r, \theta) \in \Omega \\ u(1, \theta) &= f(\theta). \end{aligned}$$

The solution was given by

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta\right),$$

where

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\, d\bar{\theta},\\
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \cos n\bar{\theta}\, d\bar{\theta} \quad \text{for } n \geq 1,\\
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \sin n\bar{\theta}\, d\bar{\theta} \quad \text{for } n \geq 1.
\end{aligned}$$

In this problem we'll use the above solution to obtain a Green's formula for u.

(a) Substitute the integrals for A_n and B_n into the formula for u, interchange the summation and the integral, and use a trigonometric identity to show that

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\theta - \bar{\theta})\right] d\bar{\theta}.$$

(b) Use the formula for the sum of a geometric series,

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad \text{when } |z| < 1,$$

to show that

$$\frac{1}{2} + \sum_{n=1}^{\infty} (re^{i\theta})^n = \frac{1 + re^{i\theta}}{2(1 - re^{i\theta})}$$

when $0 \leq r < 1$.

(c) Multiplying the numerator and denominator of the fraction in the preceding equation by $(1 - re^{-i\theta})$, show that

$$\frac{1}{2} + \sum_{n=1}^{\infty} (re^{i\theta})^n = \frac{1 - r^2 + 2ir\sin\theta}{2(1 + r^2 - 2r\cos\theta)}.$$

(d) By taking the real part of the preceding equation, show that

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1 - r^2}{2(1 + r^2 - 2r\cos\theta)},$$

and conclude that

$$u(r,\theta) = \int_{-\pi}^{\pi} f(\bar{\theta}) G(r;\theta,\bar{\theta})\, d\bar{\theta},$$

where

$$G(r;\theta,\bar{\theta}) = \frac{1}{2\pi}\left[\frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \bar{\theta})}\right].$$