Math 4163
Assignment 3

1. One easy way to reduce a PDE to an ODE is to look for solutions which depend on only one variable. You won't find the general solution this way, but you might find some important particular solutions.

Here's an example for the heat equation $u_t = ku_{xx}$.

(a) Find all solutions $u(x)$ which depend only on x and not on t. Do this by finding the general solution of an ODE. (Such solutions are called *equilibrium* solutions. They represent situations in which all temporary or *transient* fluctuations in temperature have vanished, and the temperature at a given point remains unchanged indefinitely.)

(b) Use your answer to part (a) to find the equilibrium solution to the heat equation for $0 \leq x \leq 1$, when the boundary conditions are given by

$$\begin{aligned} u(0,t) &= 2, \\ u(1,t) &= 7. \end{aligned}$$

2. For the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

for the function $u(r,t)$ (where k is a constant), derive the ordinary differential equations that are implied by the method of separation of variables. In other words, set $u(r,t) = R(r)T(t)$ and derive ordinary differential equations for the functions $R(r)$ and $T(t)$; do not forget that there should be a constant coming from the separation of variables. (Do not attempt to solve the equations you obtain.)

3. In problem (2) of Assignment 1, you found the eigenvalues and eigenfunctions for the problem

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0, \\ \frac{d\phi}{dx}(0) &= 0, \\ \phi(L) &= 0. \end{aligned}$$

It turns out that the eigenfunctions form an orthogonal system with respect to the usual inner product defined by $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. (You do not need to prove this here.) Use this fact and the method of separation of variables to find the solution of the problem

$$\begin{aligned} u_t &= ku_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ u_x(0,t) &= 0 && \text{for } t \geq 0, \\ u(L,t) &= 0 && \text{for } t \geq 0, \\ u(x,0) &= f(x) && \text{for } 0 \leq x \leq L. \end{aligned}$$

Your answer should be in the form of a series expression for $u(x,t)$ with coefficients that are given by integrals involving the given function $f(x)$. (Note: remember to take into account that the eigenfunctions you found in problem (2) of Assignment 1 probably do not form an ortho*normal* system.)