Math 4163
Assignment 8

Note: In solving these problems, if there are any steps which have already been worked out in the class lectures, you can use what we did in class without having to repeat the derivations.

1. Consider the following boundary-value problem for Laplace's equation for a function $u(r,\theta,z)$ on the cylinder $\mathcal{C} = \{(r,\theta,z) : 0 < r < a,\ -\pi < \theta < \pi,\ 0 < z < b\}$.

$$\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} &= 0 && \text{for } (r,\theta,z) \in \mathcal{C}, \\
u(r,\theta,0) &= \alpha(r,\theta) && \text{for } 0 \le r \le a \text{ and } -\pi \le \theta \le \pi, \\
u(r,\theta,b) &= 0 && \text{for } 0 \le r \le a \text{ and } -\pi \le \theta \le \pi, \\
u(a,\theta,z) &= 0 && \text{for } -\pi \le \theta \le \pi \text{ and } 0 \le z \le b.
\end{aligned}$$

Here $\alpha(r,\theta)$ is a known function.

a) Look for solutions of the problem of the form $u(r,\theta,z) = f(r)g(\theta)h(z)$. What ODEs should the functions f, g, and h satisfy?

(b) The function $g(\theta)$ should be 2π-periodic. What are the solutions of the ODE for θ that are 2π-periodic? (We've already done this part in class several times, so you can just copy the answer here.)

(c) Show that the ODE for f, after an appropriate change of variables (like the one we did in class) is Bessel's equation

$$f''(z)0(x) + \frac{1}{z}f'(z) + \left(1 - \frac{m^2}{z^2}\right) f(z) = 0.$$

Express f in terms of the Bessel functions J_m and Y_m.

(d) Use the boundary condition at $r = a$ and the condition that the limit of u exists as $r \to 0$ to obtain an explicit expression for $f(r)$. Your answer will involve z_{mn}, the nth zero of $J_m(z)$.

(e) Verify that the functions $\sinh(k(z-b))$ and $\cosh(k(z-b))$ are independent solutions of the ODE for $h(z)$ (where k is an appropriate constant). Use the boundary condition at $z = b$ to express the solution of the ODE for h in terms of one of these two.

(f) Write a series for $u(r,\theta,z)$ and use the boundary condition at $z = 0$ to find expressions for the coefficients of the series as integrals involving the function $\alpha(r,\theta)$. Use the orthogonality relation

$$\int_0^a J_m(z_{mn}r/a)J_m(z_{mn'}r/a)r\,dr = \frac{a^2 J_{m+1}(z_{mn})^2}{2}\ \delta_{nn'},$$

where $\delta_{nn'}$ is the Kronecker delta.

2. Find the solution $u(r,\theta,z)$ of the following boundary-value problem for Laplace's equation on the semi-circular cylinder $\mathcal{D} = \{(r,\theta,z) : 0 < r < a,\ 0 < \theta < \pi,\ 0 < z < b\}$.

$$\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} &= 0 && \text{for } (r,\theta,z) \in \mathcal{D}, \\
u(r,\theta,0) &= 0 && \text{for } 0 \le r \le a \text{ and } 0 \le \theta \le \pi, \\
u_z(r,\theta,b) &= 0 && \text{for } 0 \le r \le a \text{ and } 0 \le \theta \le \pi, \\
u(r,0,z) &= 0 && \text{for } 0 \le r \le a \text{ and } 0 \le z \le b, \\
u(r,\pi,z) &= 0 && \text{for } 0 \le r \le a \text{ and } 0 \le z \le b, \\
u(a,\theta,z) &= \beta(\theta,z) && \text{for } 0 \le \theta \le \pi \text{ and } 0 \le z \le b.
\end{aligned}$$

Here $\beta(\theta,z)$ is a known function. Your answer will be a series for u with coefficients which are given in terms of integrals involving $\beta(\theta,z)$.

3. Consider the following boundary-value problem for the heat equation on the interval $[0, \pi]$ with time-dependent boundary conditions:

$$
\begin{aligned}
u_t - u_{xx} &= 0 && \text{for } 0 < x < \pi \text{ and } t > 0,\\
u(0,t) &= 0 && \text{for } t \geq 0,\\
u(\pi,t) &= \pi \cos t && \text{for } t \geq 0,\\
u(x,0) &= x && \text{for } 0 \leq x \leq \pi.
\end{aligned}
$$

(a) Find a simple function $r(x,t)$ satisfying the boundary conditions $r(0,t) = 0$ and $r(\pi,t) = \pi \cos t$ for $t \geq 0$.

(b) Let $v(x,t) = u(x,t) - r(x,t)$. What differential equation, boundary conditions, and initial condition does v satisfy?

(c) Substitute $v(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin(nx)$ into the differential equation for v, and obtain an ordinary differential equation for $A_n(t)$. You may use that $x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$ for $0 < x < \pi$.

(d) Use the initial condition for v to determine $A_n(0)$.

(e) Use your answers to (c) and (d) to find $A_n(t)$. You may use that the general solution to the equation $\phi'(t) + n^2 \phi(t) = b \sin t$ is

$$\phi(t) = \frac{b}{1+n^4}(n^2 \sin t - \cos t) + Ce^{-n^2 t}.$$

(f) Use your answers to the above to write out a formula for $u(x,t)$.