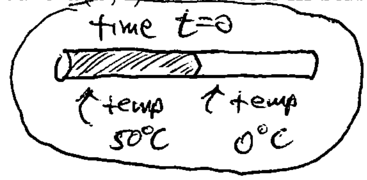
EXAM 1
Math 4163
2-17-12

Name key

1. *(20 points)* Suppose that at time $t = 0$, a rod 20 cm long has temperature of 50^o C everywhere in the left half of the rod, between $x = 0$ and $x = 10$; and temperature of 0^o C everywhere in the right half of the rod, between $x = 10$ and $x = 20$ (see diagram). The two ends of the rod are held at 0^o C for all time. The thermal diffusivity of the rod is $k = 0.5$ cm^2/s. Find the temperature $u(x, t)$ of the rod in series form. Give the coefficients B_n of the series.



The solution is given by

(5) $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$,

where $L = 20$ and $k = \frac{1}{2}$, or (2) $\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{20}\right) e^{-\left(\frac{1}{2}\right)\left(\frac{n^2\pi^2 t}{400}\right)}}$,

(3) $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{20}\int_0^{20} f(x)\sin\left(\frac{n\pi x}{20}\right)dx$, where $f(x) = \begin{cases} 50 & 0 \le x \le 10 \\ 0 & 10 \le x \le 20. \end{cases}$ (2)

So (3) $B_n = \frac{2}{20}\left[\int_0^{10} 50 \sin\left(\frac{n\pi x}{20}\right)dx + \int_{10}^{20} 0 \cdot \sin\left(\frac{n\pi x}{20}\right)dx\right] = 5\int_0^{10}\sin\left(\frac{n\pi x}{20}\right)dx =$

$= 5 \cdot \frac{20}{n\pi}\left(-\cos\left(\frac{n\pi x}{20}\right)\right)\Big|_{x=0}^{x=10} = \frac{100}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) - (-1)\right] = \boxed{\frac{100}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]}$

(5) $(n = 1, 2, 3, \ldots)$

2. *(20 points)* Consider the heat equation

$$\frac{\partial u}{\partial t} = 7\,\frac{\partial^2 u}{\partial x^2},$$

for $0 < x < 2\pi$, $t > 0$, subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(2\pi, t) = 0.$$

Solve the initial-value problem if the temperature is initially $u(x, 0) = 100 - \cos(3x)$ for $0 \le x \le 2\pi$.

The separated solutions are $u(x,t) = B\cos\left(\frac{n\pi x}{L}\right)e^{-kn^2\pi^2 t/L^2}$ (5), where

$n = 0, 1, 2, 3, \ldots$; and $k = 7$, $L = 2\pi$. Or,

$u(x,t) = B\cos\left(\frac{nx}{2}\right)e^{-7n^2\pi^2 t/(4\pi^2)} = B\cos\left(\frac{nx}{2}\right)e^{-7n^2 t/4}$ (2), (4)

with $u(x,0) = B\cos\left(\frac{nx}{2}\right)$. For $u(x,0) = 100$ we should take $n = 0$ and $B = 100$,

so $u(x,t) = 100\cos(0) \cdot e^{-0t} = 100$. For $u(x,0) = -\cos(3x)$ we should take

$B = -1$ and $n = 6$, so $u(x,t) = (-1)\cos(3x)e^{-7 \cdot 36t/4} = -\cos(3x)e^{-63t}$. (4)

So the solution is the sum of these two: (5) $\boxed{u(x,t) = 100 - \cos(3x)e^{-63t}}$.

 Name ______________________

3. *(15 points)* Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

for $0 < x < L$, with boundary conditions

$$\frac{d\phi}{dx}(0) + \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0.$$

Is $\lambda = 0$ an eigenvalue of this boundary-value problem? Carefully justify your answer.
(Note: you are not being asked to find all the eigenvalues, only whether $\lambda = 0$ is an eigenvalue.)

3

No: For $\lambda = 0$, the general solution of the differential equation is

$$\varphi(x) = B_1 + B_2 x, \quad 3$$

with $\frac{d\varphi}{dx} = B_2$ 3 for all x. So the boundary conditions require

that $0 = \frac{d\varphi}{dx}(0) + \varphi(0) = B_2 + (B_1 + B_2 \cdot 0)$, or 2 $0 = B_2 + B_1$;

and $0 = \frac{d\varphi}{dx}(L) = B_2$, or 3 $0 = B_2$.

But if $B_2 = 0$ and $B_1 + B_2 = 0$, then both B_1 and B_2 are 0, so the boundary conditions require $\varphi(x) = 0 + 0 \cdot x \equiv 0$ 2. Since the only solution for these boundary conditions is the trivial solution, $\lambda = 0$ is not an eigenvalue.

#4 ~~5.~~ *(15 points)* For the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 3u,$$

substitute $u(x,y) = h(x)\phi(y)$ and separate variables to find two ordinary differential equations for $h(x)$ and $\phi(y)$. You need not solve these equations.

Note: There is more than one way to answer this problem, depending on how you choose to separate the variables.

$$u(x,y) = h(x)\cdot\varphi(y) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{d^2 h}{dx^2}(x)\cdot\varphi(y)$$

and $\frac{\partial^2 u}{\partial y^2} = h(x)\cdot\frac{d^2\varphi}{dy^2}(y)$

and $3u = h(x)\cdot\varphi(y)$.

So the equation gives $\frac{d^2h}{dx^2}(x)\cdot\varphi(y) + h(x)\cdot\frac{d^2\varphi}{dy^2}(y) = 3\,h(x)\,\varphi(y)$. 3

Dividing by $h(x)\varphi(y)$ gives $\dfrac{\frac{d^2h}{dx^2}(x)}{h(x)} + \dfrac{\frac{d^2\varphi}{dy^2}(y)}{\varphi(y)} = 3$ 3, or

$\left(\dfrac{\frac{d^2h}{dx^2}}{h}\right) = 3 - \dfrac{\frac{d^2\varphi}{dy^2}}{\varphi}$ 3. Both sides must be constant, say "$-\lambda$"

(it doesn't matter what we call the constant). Hence $\boxed{\frac{d^2h}{dx^2} = -\lambda h}$ 3

and $3 - \dfrac{\frac{d^2\varphi}{dy^2}}{\varphi} = -\lambda \Rightarrow \boxed{\frac{d^2\varphi}{dy^2} = (3+\lambda)\varphi}$ 3

5. *(30 points)* For the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

(notice the minus sign in the equation!) for $0 < x < L$ and $y > 0$, with boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = 0, \quad \text{and} \quad u(x,0) = 0,$$

find all the possible solutions of the form $u(x,y) = h(x)\phi(y)$.

Putting $u(x,y) = h(x)\varphi(y)$ gives, after separating variables, (3)

$$\frac{\left(\frac{d^2h}{dx^2}\right)}{h} = \frac{\left(\frac{d^2\varphi}{dy^2}\right)}{\varphi} = -\lambda; \text{ (3)} \quad \text{or} \quad \frac{d^2h}{dx^2} = -\lambda h \text{ and } \frac{d^2\varphi}{dy^2} = -\lambda\varphi.$$

The boundary conditions $\frac{\partial u}{\partial x}(0,y)=0$ and $\frac{\partial u}{\partial x}(L,y)=0$ tell us that $\frac{dh}{dx}(0)=0$ and $\frac{dh}{dx}(L)=0$ (2), so $\lambda = 0$ is an eigenvalue with eigenfunction $\boxed{h(x) = B}$ (constant); (2) and, for $n = 1,2,3,\dots$; $\lambda = \frac{n^2\pi^2}{L^2}$ is an eigenvalue with eigenfunction $\boxed{h(x) = B\cos\left(\frac{n\pi x}{L}\right),}$ (2)

If $\lambda = 0$, the equation for φ becomes $\frac{d^2\varphi}{dy^2} = 0$, which has general solution $\varphi(y) = A + By$. (2) The boundary condition $u(x,0)=0$ tells us that $\varphi(0)=0$, so $0 = A + B\cdot 0$, so $A = 0$, so $\boxed{\varphi(y) = By.}$ (2)

If $\lambda = \frac{n^2\pi^2}{L^2}$, $(n = 1,2,3\dots)$ The equation for φ becomes $\frac{d^2\varphi}{dy^2} = -\frac{n^2\pi^2}{L^2}\varphi$, which has general solution (4)

$$\varphi(y) = A\cos\left(\frac{n\pi y}{L}\right) + B\sin\left(\frac{n\pi y}{L}\right).$$

The boundary condition $\varphi(0)=0$ gives us $0 = A\cos 0 + B\sin 0$, or $0 = A$, (4) so $\boxed{\varphi(y) = B\sin\left(\frac{n\pi y}{L}\right)}$ (2)

Therefore the separated solutions are

$$u(x,y) = h(x)\varphi(y) = \begin{cases} By \text{ (2)} & \left(\text{when } \lambda = 0\right) \\ \text{(2) } B\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) & \left(\text{when } \lambda = \frac{n^2\pi^2}{L^2},\ n = 1,2,3,\dots\right) \end{cases}$$